44TH TURBOMACHINERY & 31ST PUMP SYMPOSIA
HOUSTON, TEXAS | SEPTEMBER 14 – 17 2015
GEORGE R. BROWN CONVENTION CENTER

A New Equation, Using "Velocity of Sound" & "Poisson's Ratio" in the Material of Construction, Gives Quick & Remarkably Accurate Prediction Of the New Natural Frequency for Each Mode Shape of Vibration of Machinery, Components & Structures When Material & Size Are Changed

by

S.T. (Ted) Myrick, Jr., PhD

Mechanical Consultant & CEO
S.T. Myrick, Jr., PhD, LLC

(formerly)

Principal Consultant & Technology Group Leader (Retired)
Engineering Mechanics Technology Group
E. I. DuPont de Nemours Company

September 14, 2015

Dr. S.T. (Ted) Myrick, Jr., is a former Principal Consultant in the Engineering Mechanics Group for the E.I. DuPont Company. Ted holds B.S.M.E. (1967) and M.S.M.E. (1969) degrees from the University of Tennessee at Knoxville, and a Ph.D. in Mechanical Engineering from the University of Texas at Austin (1973), with emphasis on dynamics of rotor/bearing systems. Ted joined DuPont in 1973 and had assignments in Engineering R&D, Project Engineering, and the Gulf Regional Consulting Office in Beaumont, Texas. After three years of "24/7," Ted was sad to leave Texas, but happy to have a weekend again. He was named "Principal Consultant" in1990, and later became the first "Technology Group Leader" (a joint technical and management position) in the DuPont Engineering Technologies & Research Section, remaining in the Wilmington, Delaware, area for the last 20+ years of his career. Ted retired in 2010 after nearly 38 years with DuPont. He is the author of numerous technical papers, and is currently writing a technical book based on his industry experiences with mechanical equipment, covering "topics that recently-graduated mechanical engineers need to know, but … most likely do not."

In the year 2000, Ted was diagnosed with Parkinson's disease. The gradual worsening of his physical condition caused by this "affliction," and particularly, reduced stamina in the last few years, as well as, the need for medication required every few hours around the clock, made it impossible for Ted to be able to commit to the presentation schedule needed to keep the Symposium organized. The Advisory Council graciously allowed this paper to be published without being presented at the Symposium. The author, Dr. S.T. (Ted) Myrick, Jr., sincerely thanks Dr. Dara Childs, Dr. John Vance, Ms. Ashton Drollinger, (all from Texas A&M University), and Mr. Steve Locke, of the DuPont Company, for their patience and motivation.

ABSTRACT

An equation using the "velocity of sound in the material of construction," derived by the author in the mid-1970's, and published in 1979 [1], predicts with remarkable accuracy the new "natural frequencies," f_n, for all the "vibration mode shapes of interest" in a machine, a component, or a structure, when the "material of construction" and size might be changed, and as long as the materials have the same Poisson's ratio (that is; $\nu_{new} = \nu_{old.}$). That equation was:

$$f_{n\,new\,i} = (f_{n\,old\,i} / \alpha) (C_{v\,new} / C_{v\,old}), \qquad \text{Eq. 1}$$

"f_n" is the "natural frequency" for a particular mode of interest. "C_v,“ the "velocity of sound in the material of construction," is the speed at which an "impulse," or a "stress wave," travels in the material, and is defined by the equation:

$$C_v = (1/12) \sqrt{E / \rho} \quad \text{(ft /sec).} \qquad \text{Eq. 2}$$

Recently the author derived a second equation, similar to the first, but this time including a term which corrects the frequency prediction using the Poisson's ratio for the two materials if they are different. As a result, the new "Equation 1a," while still simple, can predict new natural frequencies for virtually any combination of "materials of construction." This includes metals and plastics, as well as, changes in size. "Equation 1a" is written:

$$f_{n\,new\,i} = (f_{n\,old\,i} / \alpha) (C_{v\,new} / C_{v\,old}) [(1 - \nu_{old}^2) / (1 - \nu_{new}^2)]^{1/2} \qquad \text{Eq. 1a}$$

where "α" is the "size factor" ($= L_{new} / L_{old}$,) , with "L" being a "critical dimension," such as, "length" of a cantilever plate, and "ν" (or "μ") is Poisson's ratio for the material. If the "old" and the "new" materials have the same value for Poisson's ratio, the new "natural frequency" for each vibration mode is determined by the "size factor" and the "velocity of sound in the two materials of construction," which is the original

Equation 1. And if size does not change, then α = 1, and the prediction of the "new" natural frequency" for each vibration mode becomes:

$$f_{n\ new\ i} = f_{n\ old\ i} \times (C_{v\,2} / C_{v\,1}) . \qquad \text{Eq. 1b}$$

If nothing in the design is changed other than making the machine component or structure from a different material, the "new natural frequency" for each "vibration mode shape of interest," is entirely dependent on the ratio of the "velocity of sound in each of the materials of construction." It is just that simple.

Values of the "velocity of sound in various materials" are given in examples and tables in the paper. The problem of "distortion" of a model, caused by changing some of the materials to other materials with different Poisson's ratios than those in the original design, is addressed [2, p. 641],. The original equation was derived beginning "with a blank piece of paper," using "Dimensional Analysis" and "Similitude" theory [6]. The more recent derivation of "Equation 1a" was done by re-arranging and simplifying the fundamental equation for the calculation of "natural frequencies of lateral vibration of a cantilever plate." [3, 4]. Both derivations are included in the paper. Testing conducted in the laboratory comparing "natural frequency" predictions versus measured "natural frequencies" using "shaft/disk systems" (model pump rotors), and on metal and "engineered plastic" cantilever plates (simulating axial compressor blades) showed excellent accuracy. Two examples where the equation was applied to real plant problems are also included: 1) Predicting the changes in natural frequency and stress levels in 12'-diameter sieve trays of a distillation column based on laboratory testing of a 6'-diameter aluminum model, where dozens of proposed modifications to reduce fatigue cracking and collapse of the trays were tried. Some reduce the stress; some actually made the stress worse. The trays were assembled using hundreds of bolts. A computer expert called it "just not possible" to make a computer model in the short amount of time allowed, consequently there was no computer model. 2) Predicting the "natural frequencies" of an 880-ton steel-reinforced, concrete "table" supported by 16 "springs" 35 feet in the air. The natural frequencies of the huge "spring-isolated" machinery foundation were accurately predicted from tests conducted on a 130-lbf ($1/23^{rd}$ -scale) aluminum model in our laboratory. "Equation 1" was used to define the required stiffness of small brass cantilever-beam "springs," which supported that 130-lbf aluminum model. Accuracy of the "natural frequency" predictions for the modes of vibration in all examples was excellent. The original natural frequencies required modal analysis, but no modal analysis was required after the materials were changed because the geometry was not changed. The reasoning behind that statement is included.

This paper also briefly explores the physics behind "natural frequencies," beginning with the facts that: 1) the "frequency" of free vibration is determined by the <u>material properties</u>, those being the "velocity of sound in the material," Eq. 2, and "Poisson's ratio," while, 2) the "mode shape of vibration" of a machine component or structure is determined <u>by the geomtry</u> of the machine component or structure. The paper shows that if the geometry of a part is never changed, the available modes of vibration never change. If the L/W ratio of a cantiever blade remains fixed, the mode shapes of vibration will occur in the same order and at the same frequency ratios to each other, no matter what material is used, and certain mode shapes will never be possible with that design. How the "frequency" of a "natural frequency" connects with, or becomes associated with, a particular "mode shape," or "pattern," of vibration amplitudes is discussed by the author, concluding that it seems logical it is somehow a function of a match between the "wavelength" of a frequency (or an overtone) in the bandwidth of the energy input to the system, and the geometry of an "available" mode shape of vibration. The writer recommends further exploration of the physics of "natural frequencies" with teams including engineers expert in "plate vibration" and physicists expert in "wave theory."

1.0 BACKGROUND

A Technical Need Then And Now – To Be Able To Accurately Predict The New Natural Frequencies Of Machinery Components And Structures, When Change Of The "Material Of Construction" Is Being Considered

The goal of a plant project several years ago was to create a design for "permanent" impellers to replace Teflon™-coated steel impellers for 3600-rpm, centrifugal pumps moving a "spent acid" combination that was so strong "only glass and tantalum" could hold it. The Teflon™-coated steel impellers cost a few hundred dollars each and worked well, until something sharp scratched them or otherwise caused a “pin-hole” leak in the Teflon™. When that occurred, the steel impeller “just disappeared.” The service life of the Teflon™-coated steel impellers was measured in weeks.

The project team made the new “permanent” impeller dimensionally the same as the steel impeller they had been using, but instead of steel, they made it from “Tantalum 10-W”, a material which is resistant to almost all acids. The impeller characteristics (flow capacity, head, turndown, surge points, and so on) would already be known that way. And the design team reasoned that the Teflon™ did not add any strength to the steel, so the impeller must be strong enough. There should not be any impeller dynamics issues, or the plant would have seen at least some evidence of it in pump vibration, or rubbing in the mechanical seal area behind the impeller. At least that was the apparent logic.

“Tantalum 10-W” was the material selected for the permanent impellers, but it was “very expensive,” and only enough was available to make four impellers; however, that was the “perfect number for two pumps.” The project team was given approval by management to make four impellers.

The first impeller to go into service failed catastrophically after running 14 days. It was certainly not a successful “permanent design.” The “very expensive" impeller was “a dozen or more shattered pieces in a box” when the second impeller failed catastrophically after it ran, once again, for 14 days. Each impeller operated for only two weeks. It was reported that there were no warning signs of a problem, such as high-amplitude vibration, or mechanical seal noise or distress, preceding either impeller failure.

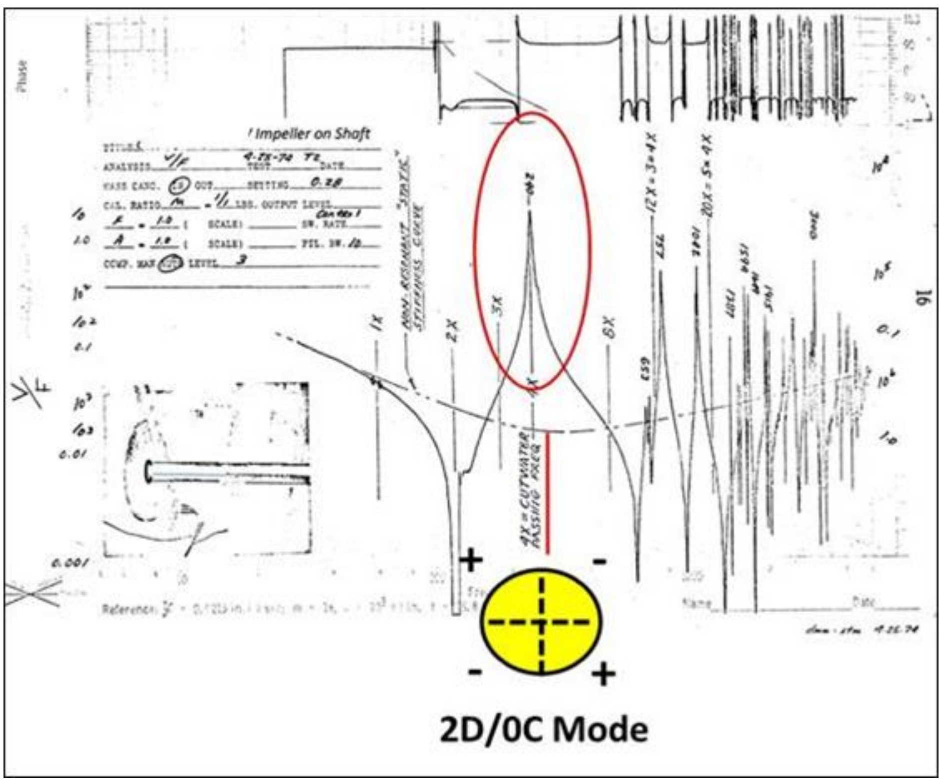


Figure 1. A “Mobility” plot done on impeller number #3, shows a very strong “2D/0C” “disk mode” of the impeller at 240 Hz.

Modal analysis of the fourth impeller was done as the third impeller was running early in “week 2” of its life. “Modal” testing identifies the natural frequencies “of interest,” and the associated “mode shapes of vibration,” for machines, machinery components, and structures.

The impellers had four vanes and ran at 3600 rpm, or 60 Hz. The "cutwater-passing" (or "vane passing") frequency, the strongest excitation force in the pump, was consequently at 240 Hz (4 vanes times 60 Hz, with one "cutwater," is 240 Hz). The primary modes of concern for impellers include the "n-diameter, zero-circular" modes – the 2D/0C, 3D/0C, 4D/0C, and so on. A sketch of the "2D/0C" mode is shown in the bottom-center of Figure 1. These modes are lightly damped, and if excited during operation, can very quickly accumulate the millions of alternating stress cycles needed to propagate a crack and cause an impeller to fail catastrophically.

The first "mobility" plot taken with a small electro-magnetic shaker attached to the impeller is shown in Figure 1. "Mobility" is the ratio of velocity output over force input (V/F). The peaks at the top of the plot are the "natural frequencies." The mode capable of the strongest resonance vibration, the 2D/0C disk mode, was found at precisely 240 Hz. That means the strongest resonant response vibration in the impeller, and the strongest excitation force in the pump, were coincident. The pump team could not possibly have been handed a worse test result.

The project goal had been to create a "permanent" impeller design. That dream evaporated. The project was a "total failure." Hundreds of thousands of dollars went down the drain. The pump team included engineers with considerable experience and knowledge in the design of pumps, but either they had not accurately known the frequency of the 2D/0C mode, or they made a serious error in judgment in proceeding with that project with that impeller design.

.

The "silver lining" for the author was that finding a solution to what appeared at that time to be a formidable problem was exactly what the author was looking for. As a brand new Engineering R&D "Research Engineer" at that time, one of my tasks was to find areas for research, with a technical need and a business willing to fund the work. The prediction of natural frequencies fit that description nicely, and it feels good to be able to report positive results.

Today, with either Equation 1 or 1a, that project team would have had far better information on which to make a decision than they had in 1974. Today, one of the steel impellers would be tested to find its natural frequencies, and then using Equation 1 or 1a, along with the "velocity of sound" in "Tantalum 10-W" and in mild steel (see Table 2), the natural frequencies of the proposed "permanent" impellers would be predicted with the following simple calculation (with $\alpha = 1$) :

$$f_{n\ new\ 2D/0C} = (f_{n\ old\ 2D/0C} / \alpha) (C_{v\ new} / C_{v\ old}) = (365 \text{ to } 370 \text{ Hz}) (1/1) (10{,}560 \text{ ft/sec} / 16{,}250 \text{ ft/sec})$$

$$= 237 \text{ to } 240.5 \text{ Hz} ,$$

which is right on the 240-Hz vane-pass, and pump "cutwater," frequency! With the 2D/0C mode in the impeller predicted to be coincident with the 240-Hz vane-pass frequency, the trouble would have been avoided by either redesigning the impeller, or just canceling the project.

The business suffered a significant financial loss in 1975, because they could not predict the natural frequencies of the modes of vibration of the impeller **before** the impellers were built. That should not happen today with Equations 1 and 1a available. This paper documents the derivation of those equations, presents examples showing how to use them and documenting their remarkable accuracy, and looks at the physics of "natural frequencies," and specifically, how a "natural" frequency becomes associated with its "mode shape of vibration."

2.0 INTRODUCTION

The Work Was Begun Nearly 40 Years Ago, But Is Still Important Today

Most of the work presented in this paper was done between 1975 and 1980, including writing the project and justifying the funding, doing the theoretical and the experimental work, and writing a "Technical Brief" showing how the original "Equation 1" came about [1]. That was 35 to 40 years, but that work is NOT "old news." It is as relevant, and important, today as it was 40 years ago. Consider that Newton's Laws were first compiled and printed by Newton in his classic book, *Principia Mathematica,* in 1687 [16], 328 years ago, and that work is not "old news" – it is still valid today, at least for velocities well below the speed of light. The results of the research was used over the forty years, but reporting the work was delayed nearly 36 years by transfers and changes in job responsibilities. The work to develop and validate Equation 1, including the derivation of the equation, was the subject of a three-page "research article" on "Scale Models for Dynamic Analysis of Machinery [1]." Published in June 1979, it was an insert in an issue of "DuPont Magazine" distributed at colleges and universities to assist recruiting. For a technical publication, the distribution was rather limited, but the article did reach a few prominent universities, from which requests for more information were received.

"Poisson's ratio" was missing from the list of variables considered in the first derivation of Equation 1. "Poisson's ratio" is the negative ratio of the lateral contraction to the axial elongation, when a uniaxial stress state is applied. Two reasons are given for the omission: 1.) Most work in our laboratory in the late 1970's dealt with metal machinery and metal structures, and metals have approximately the same Poisson's ratio (in the range from 0.3 to 0.34, so it was a non-factor, and 2.) The original equation was created using "Dimensional Analysis" theory. In "Dimensional Analysis, "the "Buckingham Pi" theorem states that: "In equivalent systems, individual parameters may vary, but "π- terms," which are <u>dimensionless,</u> are invariant [6]. Well, "π –terms" are dimensionless, and Poisson's ratio is dimensionless, so it was its own "π –term." What else can one do with it? So, Poisson's ratio was left out the first time around. This meant the original "Equation 1," when high accuracy was needed, was limited to predictions for changing from one type of metal to another type of metal, or from one plastic to another plastic. That satisfies the Poisson's ratio π-term "stringent requirement" that to "have an undistorted "true scale" model … Poisson's ratio must be the same for model and prototype structures" [2, p. 641].

We began working with "engineered plastics" in the 1990's. These were not "styrene" ("airplane-model-and-glue") plastics. "Engineered plastics" include short-glass-filled 6-6 nylon, long-glass-filled polyester, and foamed HDPE (high-density polyethylene). Many of the automotive applications were under-the-hood and metal was being replaced with plastic and "other stuff." Also, we began to see problems in steel-reinforced concrete structures, and with high-strength paperboard tubes (or "cores") used for high-speed winding of textile fibers. "Poisson's ratio" was something other than 0.3 (most metals) for a change. Plastics have Poisson's ratios in the 0.4 to 0.45 range, for high-strength steel-reinforced concrete it is 0.1 to 0.2, and for paperboard cores, Poisson's ratio varies in a "curved" pattern from 0.08 to 0.3, depending on the angle between the load and the helix angle of the wound core. We wanted to include Poisson's ratio in a new "Equation 1." But the question remained: "What else do you do with "μ"? [Note: Greek letters " ν " (nu) [3] and " μ " (mu) [4] are both used to represent "Poisson's ratio."]

As luck would have it, the answer to the question was found in one of our own labs. A poster (shown in Appendix D of this paper) described a particular test to determine the "elastic modulus," E, of a "flat plate sample of plastic" by measuring the "fundamental natural frequency" and using Eq. 3:

$$E = \frac{4\,\pi^2\,\omega_i^2\,L^4}{C_i^2} \times \frac{12\,(1-\mu^2)\,\rho}{h^2}, \qquad \text{[psi]} \qquad \text{Eq. 3}$$

Which is actually "the equation for the natural frequencies of a cantilever plate," [3] after being rearranged. The "frequency parameter," C_I, is found in tables of values in [3, 4] as a function of L/W ratio, and in Table 2 and Appendix D of this paper. It was pointed out that in Eq. 3, Poisson's ratio, μ, is in an "active" role other than that of a π-term. With that observation, "Equation 1a" was soon derived. Testing demonstrated that Equation 1a could accurately predict new natural frequencies for almost any combination of "materials of construction."

Accurately predicting natural frequencies had always seemed a formidable task, yet the equation we present as Eq. 1a in the Abstract is simple! It makes one wonder "could this really be right?" Part of the reason it appears simple is – Equation 1a is usually <u>not</u> the predictor of the natural frequencies of the original part or structure (unless that part or structure is being copied as a starting point, to be followed by a series of changes to evolve to the final design). "Equation 1a" is the predictor of the natural frequencies of the part or structure when the material or size of the original part is being considered for change. At the same time, the simple equation in most cases eliminates the need for re-doing earlier modal studies. You are predicting the new natural frequencies for the same set of modes of vibration, unless the design is “distorted” in some way, or the initial work contained errors (which a quick spot check can answer). Finally, this paper is not meant to announce or lay claim on any new scientific principal. “Natural frequency,” f_n, “shape factor” [5], “velocity of sound, Cv, in a material,” and so on, all existed before we started our work. We just never found those terms packaged in such a simple equation as was published in 1979 [1]. The accurate prediction of natural frequencies when material of construction and size are to be changed is every bit as important and as valid today as it was 35 to 40 years ago.

The most important contributions of this work might be:

1) How quick and easy it is to apply. For example, during a meeting with a manufacturer, just a minute or two is all that is needed to sort through several candidate materials.

2) No computer model is needed, which can save several critical days during a new plant startup, and

3) Once the new natural frequencies are known, the mode shapes are also known, and the modal analysis done on the original design does NOT usually have to be repeated. If we were designing cantilever plates, and we always used the same L/W ratio, we would never have to do modal analysis again.

3.0 NATURAL FREQUENCY, RESONANT VIBRATION, & MODE SHAPE OF VIBRATION

The terms “natural frequency,” “resonant vibration,” “mode shape of vibration,” and “velocity of sound in the material of construction,” used throughout this paper, may not be familiar to some readers. These terms will be briefly described here, with more detailed explanations and examples in Appendix B and C of this paper.

3.1 NATURAL FREQUENCY

If one taps a bell with a small hammer, the bell vibrates. You did not tell the bell at which frequencies to vibrate or at what amplitudes. The bell knew. The bell vibrates at one or more of its “natural frequencies.” It is “free vibration” and would continue on forever, if it were not for “damping,” a form of energy dissipation. The amplitude of vibration will depend on:
1) The amount of energy put in (“How hard did you hit it?),
2) The frequency range of the input energy (“Did you tap it with a soft rubber hammer, with a hard wood 2”x4”, the hard plastic handle of a screw-driver, or with a steel hammer?”),
3) The resonant amplification factor of the vibration mode excited (“Was it a strong, fundamental mode, or a weak, complex mode?”), and
4) Where in the vibration “mode shape” did you hit it; did you hit it on a node line?”

The ringing of a bell is more complicated than most people know.

3.2 RESONANT VIBRATION

“Resonant” vibration occurs when the frequency of the input energy is the same frequency as the “natural frequency,” and large amplitude vibration results from small input motions. A rotor “critical speed” is such a condition, where rotor speed is the same as the rotor “1st lateral natural frequency.” Figure 2a is the theoretical vibration amplitude with different amounts of system viscous damping. The high amplitude occurs as rotor speed increases passing through the “critical speed” (1st lateral natural frequency), $\omega / \omega_0 = 1.0$, and then decreasing as speed continues to increase. Figure 2b. is an actual rotor track-up, with the speed going from 1800 rpm to 10,500 rpm. To limit vibration amplitude as rotor speed passes through 3,150 rpm “1st critical speed range,” the speed is held below the “critical speed” until vibration is steady, then the steam valve is opened quickly to make the rotor accelerate rapidly through the “critical” speed range. That way, it does not “dwell” too close to the critical speed, which could allow time for vibration to build up For this to work correctly, the “lateral natural frequency” must be accurately predicted, as it was in this case.

3.3 DAMPING

During vibration, energy is dissipated in “one form or another.” “Viscous damping” is one form. It is a force proportional to velocity, which limits amplitude of oscillations at or near a “natural frequency.” If you drive a car across railroad tracks, shock absorbers limit the vertical motion of the car preventing you from hitting your head on the roof of the car. The shock absorbers employ “viscous damping.” It is the force that slows a boat when the engine stops. “Friction damping” is damping caused by one surface rubbing against another, like brake pads against the rotor. Internal damping is caused by distorting molecules. If one bounces a weight on the end of a rubber band, at the natural frequency, the amplitude can get quite large, but the weight never hits the ceiling. This is “extensional damping” caused by deformation of the rubber.

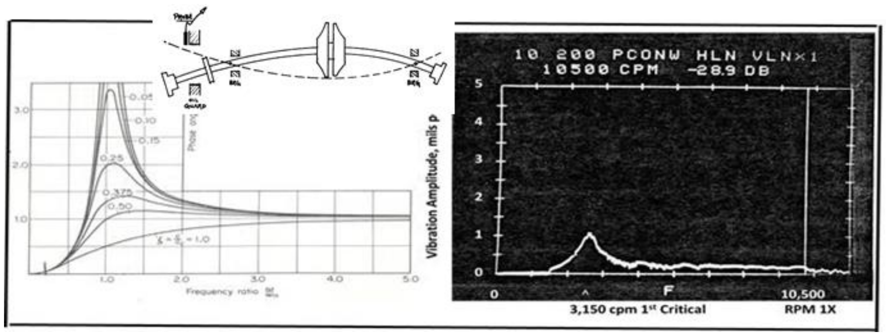


Figure 2a. Vibration of a damped rotor caused by rotating unbalance, and rotor 1st mode of with “static” unbalance.

Figure 2b. Vibration of a steam turbine passing through a “critical speed.”

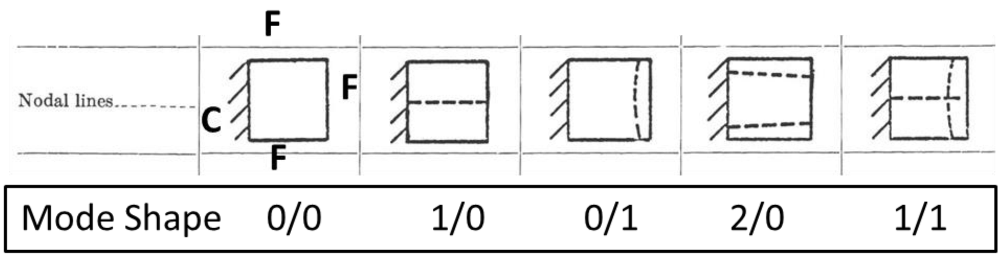


Figure 3. Mode shapes of vibration” for five different “natural frequencies” of a square cantilever plate. “Boundary conditions” are shown in the “0 / 0” mode shape – three “free”(F) edges and one edge “rigidly clamped” (that is, no lateral vibration and no angular movement on the rigid side).

3.4 BOUNDARY CONDITIONS

“Boundary conditions” (see Figure 3, mode shape “0/0”) define how and where a part is supported, and how it can move or not move. If the base of a diving board, which is rigidly “clamped” (C), is changed to a “simple support” (“pin joint,” S), the diving board becomes a pendulum. Another common boundary condition with rotors is “free-free” (F), where the rotor is hung from a very soft “bungee cord,” rather than supported in its bearings.

3.5 "VELOCITY OF SOUND IN A MATERIAL," C_v

It is just what it sounds like: The speed at which sound travels in a given material. It is also the speed at which a stress wave, or an impulse, travels in a material.

Sound travels in dry air (32° F; 0 °C) at 1,088 ft/sec, and sound travels through water at 4,794 ft/second. Sound travels much faster in metal. Take as an example, the "velocity of sound" in "aluminum."

The velocity of sound, C_v, in a material of construction, is defined by Eq. 2 as:

$$C_v = \sqrt{E / \rho} \quad (\text{in / sec}), \quad \text{or} \quad (1/12)\sqrt{E/\rho} \quad (\text{ft /sec}) \qquad \text{Eq. 2a , 2b}$$

where E = elastic modulus (lbf / in^2), and

$$\rho = \text{density} = \gamma / g = \text{specific weight} / 386 \text{ in/sec}^2 \qquad \text{Eq. 2c}$$

with units of : $(\text{lbf / in}^3) / (\text{in/sec}^2) = \text{lbf-sec}^2 / \text{in}^4$.

The specific weight and elastic modulus, along with the gravity constant, and Eq. 2, are all one needs to determine the "velocity of sound in a material." [Note: The term "of construction" is often added to differentiate a material like "water" from a material, such as titanium or aluminum, from which a machine component, or structure, might be built, and then it is: "The velocity of sound in a material of construction."]

For "aluminum" --

E = 10.00 E+06 lbf/in^2 Sp Wt = 0.100 lbf/in^3

Density = ρ = 0.100 lbf / in^3 / 386 in/sce^2 = 2.59 E-04 lbf -sec^2 / in^4.

So, then, substituting into Eq. 2 :

$\mathbf{C_v}$ = [10.00 E+06 lbf /in^2 / 2.59 E-04 lbf sec^2 / in^4]$^{1/2}$

C_v = 196,469 in/sec , the "velocity of sound in aluminum," [w/ units of: "inches / second"].

Or

C_v = (196,469 in/sec) / (12 inches / ft)

C_v = 16,375 ft / sec = the velocity of sound in "aluminum" [w/ units of: "ft / sec"]

The "velocity of sound in aluminum" is 15 times faster than the velocity of sound in dry air.

3.6 IF YOU HAD TWO BELLS

1) That Are Geometrically Identical Down To The Tiniest Detail

2) But Are Made Of Two <u>Different</u> "Materials Of Construction"

If you had two bells which were exactly identical in size, dimension, and even the design on the surface, but one was made of "bronze" and the other was made of "titanium", would the "mode shapes of vibration" be the same, or different, in the two bells? Would they naturally ring at the same or at different frequencies if they were tapped with a small hammer? If you knew the "natural frequencies" and "mode shapes of vibration" for one bell, how would you determine the "natural frequencies" and "mode shapes" for the other bell, and how long in time do you estimate it would take?

3.7 WHAT IF YOU HAD TWO AXIAL COMPRESSOR BLADES, INSTEAD?

1) Geometrically Identical Down To The Tiniest Detail

2) Made Of Two Different "Materials Of Construction"

If there were two axial compressor blades exactly identical in size -- ALL the dimensions, even the surface finish -- but one was made of "bronze" and the other was made of "titanium", would the "bell" questions be easier to answer? Read Section 3.8, let the blades be modeled as "cantilever plates" with height-width ratios of 1.5:1 and 3.0:1, and see if you can answer the questions.

3.8 THE ROLES OF "MATERIAL" AND "GEOMETRY" IN PREDICTING "NATURAL FREQUENCIES

Axial compressor blades can be thought of as "cantilever plates" to answer the "natural frequency" questions posed. Equation 4 is the equation for calculating the "lateral natural frequencies of cantilever plates" [3]:

$$\omega_i = \frac{C_i\, h}{2\pi L^2} \sqrt{\frac{E}{12\,(1-\mu^2)\,\rho}} \quad , \qquad \text{Hz} \qquad\qquad \text{Eq. 4}$$

The "frequency parameter," C_i, is found in tables of values in Ref's [3, 4] as a function of plate L/W ratio, and in Table 1 of this paper. Nothing obvious pops out from looking at Eq. 4 or at the C_i table which might help answer the "bell" questions; however, when the C_i values from Table 1 are plotted versus the L/W-ratio of the cantilever plate, as it is in Figure 4, the role of the "L/W" ratio becomes apparent.

Plate Mode	First	Second	Third	Fourth	Fifth
Mode Shape	0/0	1/0	0/1	2/0	1/1
L/W	C_1	C_2	C_3	C_4	C_5
0.5	3.508	5.372	21.96	10.26	24.85
1	3.494	8.547	21.44	27.46	31.17
1.5	3.482	11.6			
2	3.472	14.93	21.61	94.49	48.71
2.5	3.465				
3	3.460				
3.5	3.456				
4	3.453				
4.5	3.451				
5	3.450	34.73	21.52	563.9	105.9

Table 1. "Frequency parameters," C_i vs L/W ratio for a "cantilever plate."

Select a length-to-width (L/W) ratio. Choose L/W of 1.5, for example. Read from left to right in Table 1, and from the bottom up in Figure 4, as if a sweep oscillator was sweeping a shaker through a frequency range in a modal test. Since ω_i is proportional to C_i, the vertical axis in Figure 4 can also be thought of as frequency, ω. The first coefficient you see is C_1 -- the 1st "plate mode," which is also the 1st beam bending mode. Then as the shaker frequency increases, the 2nd plate mode would be excited, then the 3rd plate mode (also called the 2nd beam bending mode), and the 3rd bending and 4th plate modes would form a complex mode, since they have the same frequency.

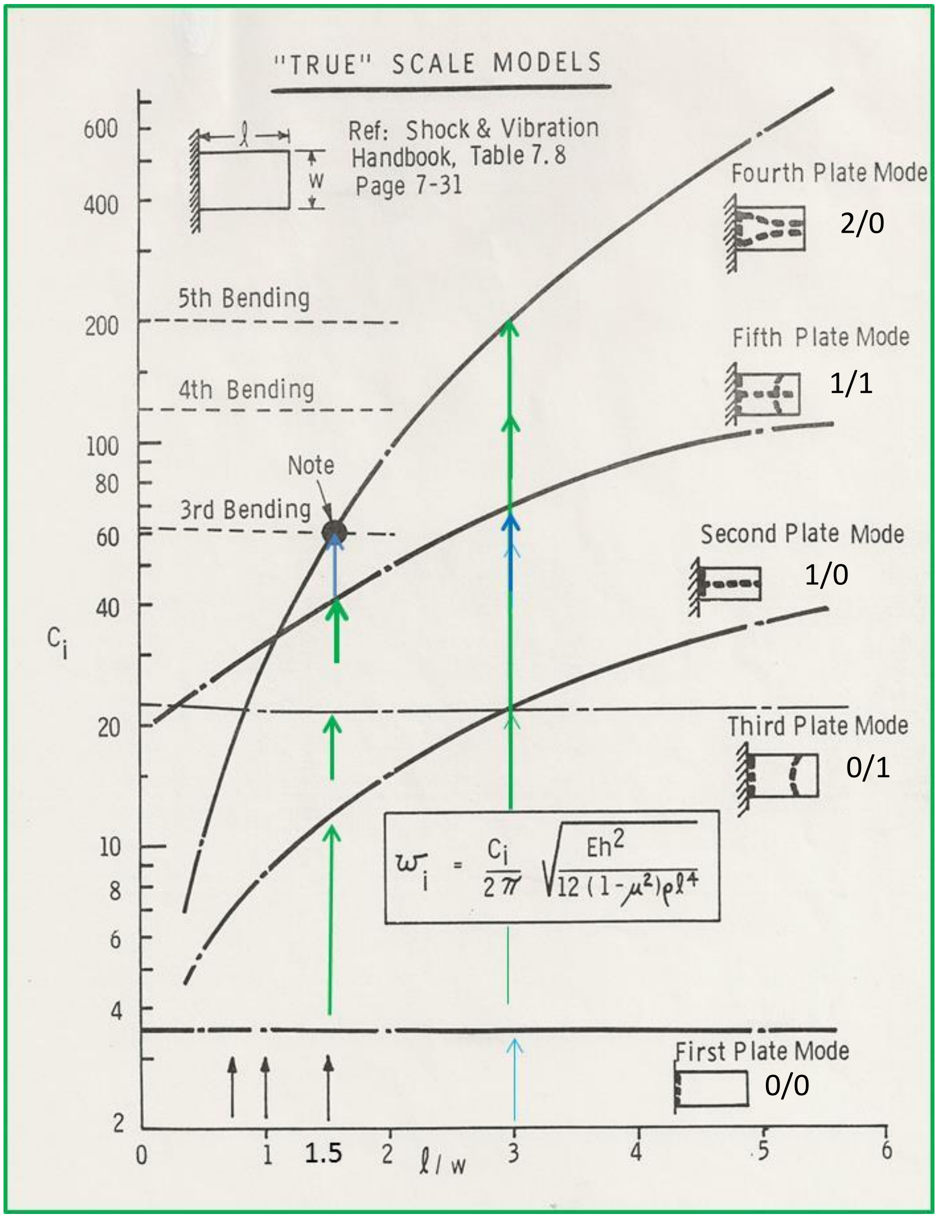


Figure 4. Plot of C_i "frequency parameters," vs L / W ratio for a "cantilever plate."

Next, do the same "sweep frequency" test for a cantilever beam with a L/W ratio this time of 3.0. Again, the 1st mode excited would be the "fundamental beam bending" mode or the "1st plate mode." But then, as vibration frequency increased, the vibration spectrum being generated would look totally different from the spectrum generated with the L/W ratio of 1.5. With the L/W = 3 plate, a complex mode with the 2nd and 3rd plate modes is evident, followed by the 3rd beam bending mode which is its self followed closely by the 5th plate mode. That is not anywhere near the "order of appearance" – "the mode "sequence" – of the beam with the L/W ratio of 1.5.

This is what happens when a "model" is "distorted." The modes for a particular L/W ratio do not occur in the model in the same order as in the original part. And predicting the new "modes of vibration" then becomes impossible!

But if the L/W ratio remains the same, the "mode shapes of vibration" will always occur in the same order and at the same frequency ratio to each other (for example, C2:C1, C3:C1, and so on), making it possible to accurately predict the new "natural frequency" for each "mode shape of vibration."

In this example, the order of appearance of mode shapes is now known for the first five modes of vibration of a cantilever plate for two different L/W ratios – or of a cantilever blade in an axial air compressor, as another example – however, "material" has not been mentioned. Change the material and the frequency will change, but the order in which the various modes of vibration appear, and the ratios of the natural frequencies to each other, will remain the same. As long as geometry stays the same, modal analysis does not have to be redone!

Summarizing, frequency changes if "material" changes, but "mode shape" is a function of geometry. How many of the "bell" questions can be answered now? And how, physically, does each natural frequency become associated with (or choose, or connect with) one particular mode shape of vibration?

4.0 DERIVATION OF "EQUATION 1"

Using "Similitude" & "Dimensional Analysis" To Derive "Equation 1" For The Case Where Poisson's Ratios Are The Same ($\nu_1 = \nu_2$)

"Equation 1" was derived beginning with two cantilever beams (Figure 5), one named the "model" and the other the "prototype." Using "Dimensional Analysis," and "Similitude theory [6], the equations just fell right in place. The derivation is presented here because it is fairly short, and if one uses an equation, it is important to understand the "ground rules" under which that equation was derived, so you know its limitations.

A cantilever beam may be used to derive the expressions needed to predict natural frequencies and vibration modes for prototypes made entirely of one material. Refer to Figure 5. The larger cantilever beam is the "prototype" (or "new") beam. The small one is the "scale model" (or the "original") beam. A "scale model" implies a change in one or more physical parameters between model and prototype – size, material, or both. In the "true" scale model the "scale factor", α, is a model where all of the dimensions have been changed by the same ratio. Also, boundary conditions and fabrication methods are the same. There is an example of both a "true" scale model and a "slightly distorted" scale model (a tall, thin blade, where the critical dimension is the length of the blade) later in the paper.

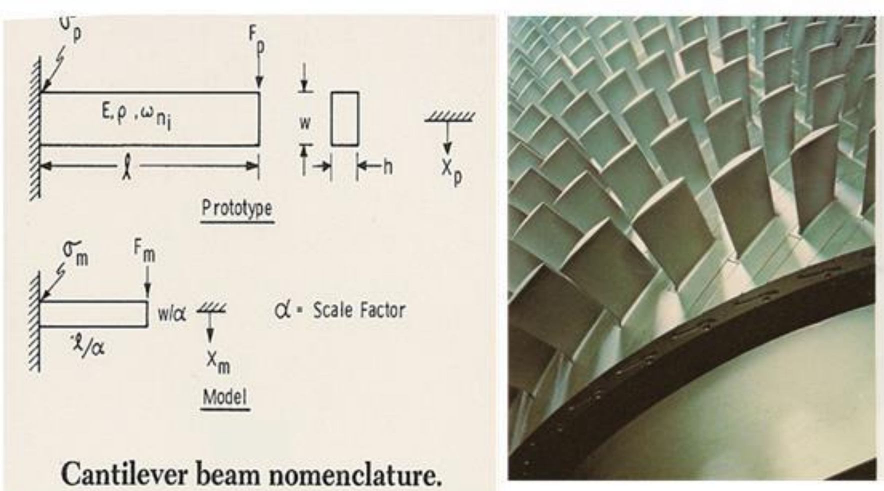


Figure 5. Large and small cantilever beams (the "prototype" and the "model") are simple models of cantilever blades in axial compressors.

In “dimensional analysis,” one begins with the Buckingham Pi theorem: “In equivalent systems, individual parameters may vary, but “π terms,” which are dimensionless, are invariant” [6]. The number of π terms needed to describe a system may be found from:

$$n = q - b \qquad \text{Eq. 5}$$

where “n” is the number of π terms, “q” is the total number of parameters needed to describe the system, and “b” is the number of basic dimensions involved.

The total number of parameters needed, “q”, is 6 : (1) length, l, (2) width, w, (3) thickness, h,

4) Young’s Modulus, E, (5) density, ρ, and (6) natural frequency, ω_i ,

where “i” indicates a particular mode of vibration.

Three basic dimensions are involved:

(1) inches, (2) pounds-force (lbf), and (3) seconds.

These come from:

length, L (inches), elastic modulus, E (lbf/in^2), and density, ρ (lbf-sec^2/in^4),

which is:

$$\rho = \text{specific weight divided by the gravity constant} = w' / g_c \qquad \text{Eq. 2c}$$

$$= w' \, [\text{ lbf/in}^3 \,] \; / \; 386 \;\; [\text{ in/sec}^2 \,].$$

So, “b” = 3.

Then the number of “π terms” is three: $n = 6 – 3 = 3$ and the “π terms” are:

$$\pi_1 = h / \ell = \text{thickness / length} = \text{inches / inches} = \text{dimensionless} \qquad \text{Eq. 6a}$$

$$\pi_2 = w / \ell = \text{width / length} =\backslash \text{ inches / inches} = \text{dimensionless} \qquad \text{Eq. 6b}$$

and rearranging the remaining variables until the term is dimensionless:

$$\pi_3 = (\omega_i \times \ell) / (E / \rho)^{1/2} \qquad \text{Eq. 6c}$$

$$= (\text{rad /sec}) \times (\text{ inches}) / [\text{ lbf /in}^2 / (\text{ lbf-sec}^2 / \text{in}^4)]^{½}$$

$$= (\text{in / sec}) / [\text{ lbf / in}^2 \times (\text{in}^4 / \text{lbf-sec}^2)]^{½} = (\text{in/sec}) / (\text{in / sec}) = \text{dimensionless.}$$

Knowing that C_v , the “velocity of sound” in a material, is

$$C_v = \sqrt{E / \rho} \qquad \text{Eq. 2a}$$

The “velocity of sound” in a material is briefly explained in Section 3.6, and tables of values are presented Tables 2 and 7, and in Appendix C. The third “π term” becomes:

$$\pi_3 = (\omega_i \times \ell) / (E / \rho)^{1/2} = \omega_i \times \ell / C_v. \qquad \text{Eq. 6d}$$

For a “true scale model,” where all the dimensions are scaled by the same factor, α,

$\pi_{1p} = \pi_{1m}$ and $\pi_{2p} = \pi_{2m}$, Eq. 7a and 7b

Where "p" is the "prototype" and "m" is the "model."

From ***"the principal of equivalent systems,"*** ***π_{3p} must equal π_{3m}.***

Equating the π_3 terms:

$$\pi_{3p} = (\omega_{ip} \times \ell_p) / C_{v\text{-}p} = \pi_{3m} = (\omega_{im} \times \ell_m) / C_{v\text{-}m} \qquad \text{Eq. 7c}$$

Solving for the natural frequency of the "prototype" (that is, the "new" part) yields:

$$\omega_{ip} = \omega_{im} \times (\ell_m / \ell_p) \times (C_{v\text{-}p} / C_{v\text{-}m}) \qquad \text{Eq.76d}$$

Substituting the "scale factor" –

$$\alpha = \ell_p / \ell_m \qquad \text{Eq. 8}$$

yields the equation we call "Equation 1" – the one we were looking for:

$$\boldsymbol{\omega_{ip} = (\omega_{im} / \alpha)(C_{v\text{-}p} / C_{v\text{-}m})} \qquad \text{Eq. 1}$$

Which reads: The natural frequencies of machine components and mechanical structures, made of the same type of material (e.g., metal model and prototype, or plastic model and prototype, etc.) is determined by the ratio of the velocities of sound in the materials, and a size factor. Simple!

4.1 $C_{v\text{-}I}$ -- THE "VELOCITY OF SOUND, C_v, IN A MATERIAL (I)"

A few values for the "velocity of sound" in materials commonly used in petrochemical plant machinery can be found in Table 2. Others can be found in Table 7 and Appendix C in this paper. These velocities calculated (except air and water) using E and ρ in Eq. 2c.

Material	Velocity of Sound	Material	Velocity of Sound
Aluminum (aircraft)	16,375 ft/sec	Magnesium	16,250 ft/sec
Brass (70 – 30)	11,632	Mild Carbon Steel	16,250
Concrete (w steel rebar)	11,825	Stainless Steel 316- L	16,100
Inconel 600	16,725	Steel 4340	16,850
Inconel 625	16,185	Tantalum 10W	10,560
Hastelloy C-276	15,775	Titanium	16,625
Air (Dry C)	1,088	Water	4,794

Table 2. Velocity of Sound, C_v, in Various Materials at Room Temperature.

4.2 EXAMPLE 1: PREDICTING NEW NATURAL FREQUENCIES –

Metal Blade Of A New Material To Replace A Metal Blade Of A Different Material

A stationery inlet guide vane in an axial compressor suffered a fatigue failure during operation. The blade that failed was 316-L SST (that is, 316-L stainless steel). The manufacturer proposed to change the guide vane material of construction to "titanium."

The dimensions would remain the same in the proposed blades, so $\alpha = 1$. Substituting into Equation 1:

C_v for stainless steel is16,100 ft/sec and for titanium 16,625 ft/sec. .

$$\omega_{\text{titanium}} = (\omega_{sst} / 1)\,(C_{v\text{-titanium}} / C_{v\text{-sst}}) = (\omega_{sst})\,(16{,}625 \text{ fps} / 16{,}100 \text{ fps}) = 1.0326\ \omega_{sst} .$$

Equation 1 predicts the natural frequencies of the new blades would be 3.26% higher in frequency. The modes of vibration would appear with the titanium blades in the same order as they occur in the stainless steel blades, since modes are governed by the geometry, and $\alpha = 1$. Material damping (or "loss factor") is low enough with all the materials listed in Table 1, for Equation 1 to be very accurate.

4.3 VALIDATION AND ACCURACY OF "EQUATION 1"

We used, instead of bells, four "pump rotors" made of different materials and size. Figure 6 shows four shaft/disk simulated rotors of various sizes with three being made of mild steel and one from aluminum. The data presented here is for when the half-size steel rotor was tested for natural frequencies and mode shapes, and the natural frequencies of the first twelve modes were used to predict the first twelve modes of a full-size aluminum rotor using "Equation 1," where the C_v values for steel and aluminum were taken from Table 2. The actual frequencies were incredibly close to the predicted frequencies for the various modes. Eight frequencies were less than 1% different, and the maximum difference was less than 2% case (see Table 3).

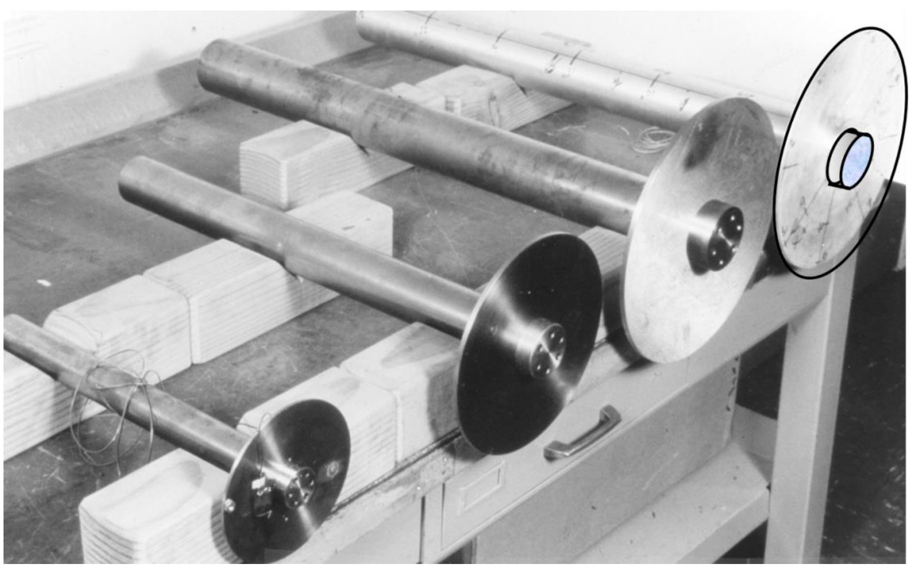


Figure 6. Four simulated rotors were used to study model "distortion." The steel rotors were full-sized, three-quarter size, and half-size.The aluminum rotor was also "full-sized."

The form of Equation 1 for this test was: $f_{n\text{-alum}} = (f_{n\text{ steel}} / 2)(16{,}375 \text{ ft/sec} / 16{,}250 \text{ ft/sec})$

$$f_{n\text{-alum}} = 0.504\ f_{n\text{ steel}}$$

Mode of Vibration [Number of diametral (D) and circular (C) nodes]	Full-Scale Aluminum Prototype Measured Hz	From Half-Scale Steel Model Predicted Hz	△ %
1D/0C	324 - 326	320	1.5
0D/1C	372 - 379	373	0.0
2D/0C	439	439	0.0
1st Shaft	497 - 502	490	1.4 to 2.4
3D/0C	832 - 833	830	0.3
2nd Shaft	1336	1314	1.6
4D/0C	1428 - 1430	1438	0.6
5D/0C	2173 - 2189	2180	0.0
1D/1C	2330 - 2337	2320	0.4 to 0.7
3rd Shaft	2598 - 2600	2563	1.4
2D/1C	2832 - 2840	2815	0.6 to 0.9
6D/0C	3078 - 3080	3086	0.2

Table 3. Test results for natural frequency prediction from a half-size steel "model" to a full-sized aluminum "prototype."

4.4 REQUIREMENTS FOR ACCURACY:

In testing done in our lab and in our plants, we confirmed "Equation 1" to be highly accurate for all sorts of complex structures, as long as:

- All the dimensions are changed by the ratio, α,
- The same fabrication methods will be used,
- Fundamentally it is made of one material, with fairly low damping, and
- The structures are supported in the same way.

The natural frequencies, subscript "i", will be in the same order in the new structure, and the mode shapes will be faithfully reproduced.

4.5 IMPROVED ACCURACY OF ACCELEROMETER MEASUREMENT

Accuracy of the Young's Modulus test depends on determining the correct frequency for the vibration of the cantilever plate. But if you attached an accelerometer to the beam, the frequency you measured will be lower than if the accelerometer is not attached. The mass and the rotatory inertia of the accelerometer are included in the vibrating mass.

Figure 7 shows a quick method for determining the error caused by the accelerometer, and eliminating the error by extrapolating. Repeat the measurement with the accelerometer located closer and closer to the base of the cantilever beam. The correct value is that frequency when the accelerometer is at the base of the beam.

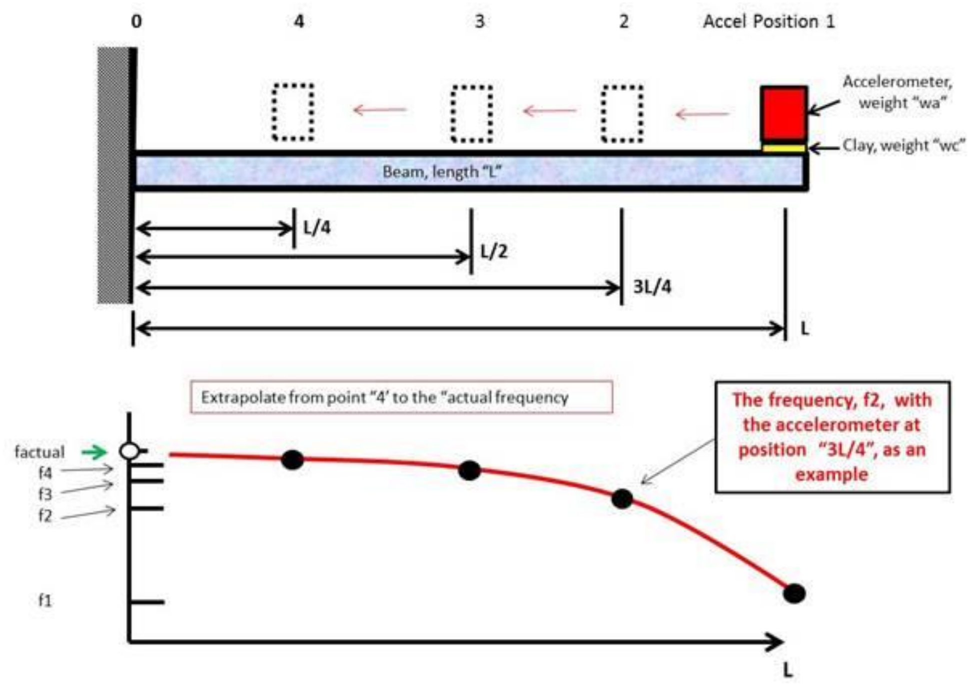


Figure 7. A quick method for determining the error caused by the accelerometer and eliminating the error by extrapolating to find the natural frequency of a blade, for example, without the accelerometer weight being involved

4.6 WHAT ABOUT A "SLIGHTLY" DISTORTED MODELS

If you change too many dimensional relationships, the "π terms," then you no longer have a valid model. If you put an aluminum disk on a steel shaft, and try to predict the frequencies of a steel disk on an aluminum shaft, it becomes un-workable. This is shown by the mode shape interactions shown in the "mode shape" map in Figure 4.

But two "distortion" problems we found we could handle were: 1) the "n"D/0C disk (impeller) modes of pumps and small steam turbines, where there is almost no interaction between the shaft and the disk, consequently, the shaft material does not make much difference, and 2) a "slightly distorted" model where **both the material and the height** of an axial compressor blade were **changed** at the same time (see Figure 8).

In the case of "n"D/0C disk modes, because the impeller motion is predominantly axial, and the vibration "mode shapes" show the impeller segments are vibrating in "equal and opposite motions," (two quadrants are moving forward, while two quadrants are moving backward), there is almost no interaction of the "n"D/0C "disk modes" with the shaft where the disk is shrunk on the shaft. Consequently, these modes are extremely lightly damped. They really only have "internal material" damping, also called "solid" damping, and "viscous" damping from water droplets in the steam. "Internal material" damping can be quantified in terms of a "loss factor," γ , using the "half-power" method [7]. In tests conducted at a turbine repair shop, the bladed disks in small one- and two-stage steam turbines commonly used in plant power houses to drive boiler feedwater pumps, were lightly impacted with the hard-plastic handle of a large screwdriver. "Amplification factor," Q , (= $1/\gamma$), of over 40:1 were measured, and the vibration of the 2D/0C mode could still be detected with an accelerometer and Fourier analyzer three to four minutes after the disk was impacted.

In the case of blade material and height changes, in just one or two minutes one can determine that all the frequencies will be within 0.15% of where they were with stainless steel!

The approach is to address the changes separately: (a) First change the material assuming a scale factor of α =1, then (b) use the cantilever beam natural frequency (Equation 4) to determine how much the change in height will change the frequency in the blade with the new material. This is illustrated in Figure 8.

- First, keeping the dimensions the same ($\alpha = 1$, a "true" model), change the material and get the new natural frequencies as if the blade dimensions were not to be changed. Use Equation 1 for this:

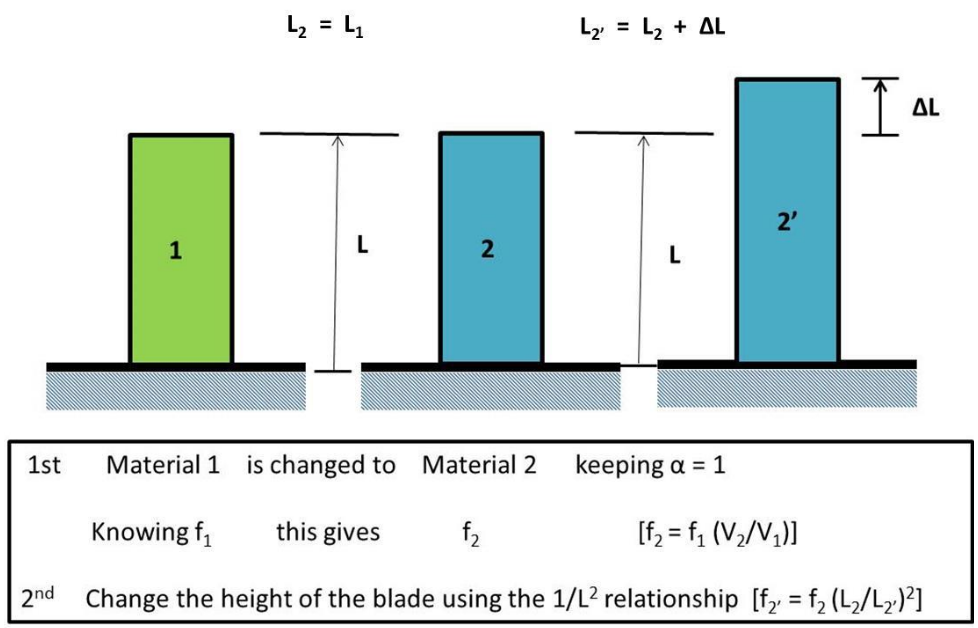


Figure 8. How to handle a "slightly-distorted" model.

$$f_{tit'} = f_{sst} (C_{v\text{-}tit} / C_{v\text{-}sst}) \qquad \text{Eq. 1}$$

$$f_{tit'} = f_{sst} (16{,}625 \text{ ft/sec} / 16{,}100 \text{ ft/sec})$$

$$f_{n\text{-}tit'} = 1.0326 \; f_{n\text{-}sst}$$

Then, let the blade get taller. This is a "distorted model", so α does not equal L_2 / L_1. The first mode of the new blade (3.625 in. tall versus the original blade height of 3.5625 in.) becomes:

$$f_{2'} = f_2 (L_2 / L_{2'})2 \qquad \text{Eq. 9}$$

$$= f_{tit'} \; (3.5625 \text{ in} / 3.625 \text{ in})^2 = 0.9658 \; f_{tit'}$$

The change to titanium moves the frequencies up by 3.26%, and the 1/16" increase in height lowers the frequency by 3.4%. So after all the changes are made, the blade natural frequencies will be <u>almost exactly where they were</u> with the original blade material (1.036 x 0.9658 = 0.9973) and height. The frequency will be almost unchanged!

Equation 9 came from the equation for the "natural frequency of a cantilever plate" (or blade) found in [3]. How "f_n" varies with "L" is shown below:

$$\omega_i = \frac{C_i h}{2\pi L^2} \sqrt{\frac{E}{12\,(1-\mu^2)\,\rho}} \quad , \text{ Hz} \qquad \text{Eq. 4}$$

which after multiplying both sides by L^2 can be written

$$\omega_i L^2 = \frac{C_i h}{2\pi} \sqrt{\frac{E}{12\,(1-\mu^2)\,\rho}} \quad , \text{ Hz} \qquad \text{Eq. 4'}$$

The square root of the terms on the right in Eq. 4' are the same for any $\omega_i L^2$, so equating the square root term gives:

$\omega_{i\,1} L_1^2 = \omega_{i\,2} L_2^2$ = constant, and then ratioing $\omega 1$ to $\omega 2$ gives

$\omega_{i\,2} = \omega_{i\,1} (L_1^2 / L_2^2)$, which shows where the equation above originated. Eq. 10

With that lead in to what can be done rearranging the cantilever equation --

5.0 DERIVATION 2: "SIMPLIFYING" THE EQUATION FOR THE "NATURAL FREQUENCIES OF LATERAL VIBRATION OF CANTILEVER PLATES" (EQ. 4)

The equation for "the natural frequencies of lateral vibration of a cantilever plate," Eq. 4," is:

$$\omega_i = \frac{C_i h}{2\pi L^2} \sqrt{\frac{E}{12\,(1-\nu^2)\,\rho}} \quad , \text{ Hz} \qquad \text{Eq. 4}$$

For this mathematical exercise, the equation was generated by combining the "Flexural Rigidity" and "Frequency Parameter" terms from Chapter 7 of the <u>SHOCK &VIBRATION HANDBOOK, 3rd Ed</u>, Chapter 7 [3], because these terms "made sense." The same equation can be generated using slightly different "Flexural Rigidity" and "Frequency Parameter" terms found in the book <u>Vibration of Plates</u>, by Leissa [4], but that derivation is not for the novice. Leissa uses density as "mass density per square area" and a "frequency parameter" that appears to be minus an "h" needed for the units to match those in Leissa's "density per square area." Also, Leissa [4] and the SEM Handbook [2] use "ν" for Poisson's ratio, whereas, the S&V Handbook uses "μ" for Poisson's ratio [4]. You have to pay special attention to units in mechanical engineering calculations.

So, beginning with these terms from "Classical Lateral Plate Vibration Theory," the two terms are

"Flexural Rigidity", $D = E h^3 / 12 (1-\nu^2)$ Eq. 11a , and

"Frequency Parameter" $\omega_n / \sqrt{D g / \gamma h a^4} = C_i$ Eq. 11b.

C_i is the “Frequency Parameter” found in Figure 4, and as explained earlier, it is proportional to the frequency of vibration of that mode. “Mode shapes for our purposes are sketches, or small maps, showing the “node lines” in the part at that frequency. “Node lines” are lines which have zero lateral vibration amplitude, but they do allow (and in most cases, “require”) rotation of the plate along those lines.

In Eq. 4, “E” is Young’s modulus (lbf / ins),” h” is the plate thickness, (inches) ,“a” is the length of the cantilever plate (inches).and “$\gamma / g_{C,}$” which is the “weight density,” or the “specific weight” divided by “g_C”, the “gravity constant” (386 in / sec^2) .

Note in Figure 9 below, from Leissa’s book [4] that the “frequency parameter” is different than the “frequency parameter” used by Harris in the tables in Shock and Vibration (Eq. 11b). This is because Leissa’s book uses density “ρ” defined as “mass density <u>per unit area</u> of the plate,” where Harris’ tables have the extra “h” needed in the denominator and, “weight density “ divided by “g_C”, The units conversions in “classical lateral plate deflection theory” are not for the novice.

Referring now back to equations 11a and 11b, substituting “ D” (Eq. 11a) into Eq. 11b, and rearranging terms yields the equation for the “natural frequencies of cantilever plates;”

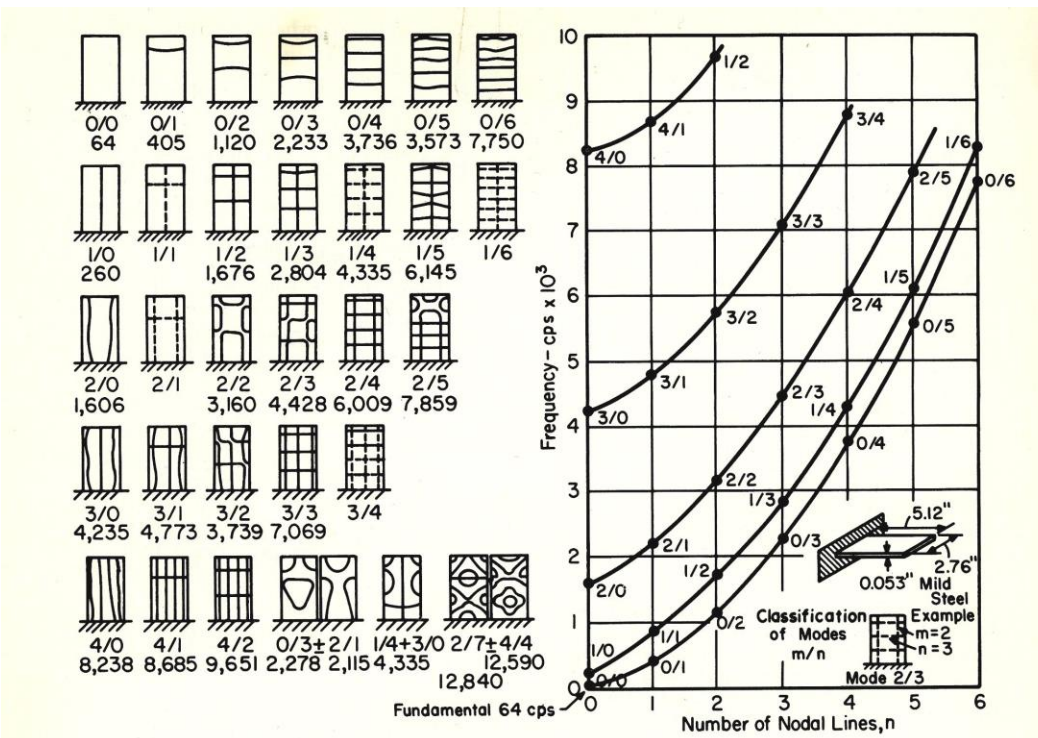


Figure 9. Various “mode shapes of vibration” for cantilever plates on the left side, and a plot of the “frequency coefficient,” C_i , on the right versus number of nodal lines for the various modes. The frequency increase as the modes ecomes more complex.

The equation for natural frequency of any mode of vibration of a uniform thickness “cantilever plate”, substituting “L” for plate length “a” --

$$\omega_i = \frac{C_i\, h}{2\pi L^2} \sqrt{\frac{E}{12\,(1 - \nu^2)\,\rho}} \quad , \text{ Hz} \qquad \text{Eq. 4}$$

So now, do you remember back to the days of algebra and square roots?

$$\sqrt{36} \quad = \quad 6$$

but also, subdividing the square root gives the same answer.

$$\sqrt{36} \quad = \quad \sqrt{9} \quad \sqrt{4} \quad = \quad (3)(2) \quad = \quad 6$$

So, we will use this approach to simplify Equation 4.

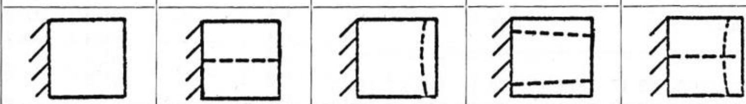
Mode	1	2	3	4	5
$\omega a^2\sqrt{\rho/D}$	3. 494	8. 547	21. 44	27. 46	31. 17
Nodal lines					
Amplitude coefficients.	$A_{11}=$ 1. 0000 $A_{13}=-0.$ 0087 $A_{15}=-0.$ 0008 $A_{21}=-0.$ 0026 $A_{23}=-0.$ 0050 $A_{25}=-0.$ 0011 $A_{31}=$ 0. 0001 $A_{33}=-0.$ 0014 $A_{35}=-0.$ 0006	$A_{12}=$ 1. 0000 $A_{14}=-0.$ 0134 $A_{16}=-0.$ 0011 $A_{22}=$ 0. 1212 $A_{24}=$ 0. 0044 $A_{26}=$ 0. 0006 $A_{32}=-0.$ 0020 $A_{34}=-0.$ 0011 $A_{36}=-0.$ 0006	$A_{11}=$ 0. 0054 $A_{13}=$ 0. 2731 $A_{15}=$ 0. 0092 $A_{21}=$ 1. 0000 $A_{23}=$ 0. 0713 $A_{25}=$ 0. 0079 $A_{31}=-0.$ 0118 $A_{33}=$ 0. 0050 $A_{35}=-0.$ 0003	$A_{11}=$ 0. 0090 $A_{13}=$ 1. 0000 $A_{15}=-0.$ 0120 $A_{21}=-0.$ 2866 $A_{23}=$ 0. 1786 $A_{25}=$ 0. 0009 $A_{31}=-0.$ 0451 $A_{33}=$ 0. 0125 $A_{35}=-0.$ 0023	$A_{12}=-0.$ 1201 $A_{14}=$ 0. 0627 $A_{16}=$ 0. 0080 $A_{22}=$ 1. 0000 $A_{24}=-0.$ 0388 $A_{26}=-0.$ 0013 $A_{32}=$ 0. 0776 $A_{34}=$ 0. 0086 $A_{36}=$ 0. 0024

Figure 10. Table 4.45 from Leissa for square cantilever plate.

Noting that cantilever plate width is "b" and "h" is thickness, then subdivide the terms under the square-root symbol:

$$\omega_i = \frac{C_i h}{2\pi L^2}\sqrt{\frac{E}{12(1-\nu^2)\rho}} = \left(\frac{Ci}{2\pi}\right)\left(\frac{h}{L^2}\right)\sqrt{\frac{1}{12}}\sqrt{\frac{1}{(1-\nu^2)}}\sqrt{\frac{E}{\rho}}$$

Eq. 11

By the definition of "Velocity of Sound", C_V , of a material is

$$\mathbf{C_V = \sqrt{E/\rho}} \quad , \qquad \text{(in / sec.).}$$

Eq. 2a

Then equation (11) becomes

$$\omega_i = \frac{C_i h}{2\pi L^2}\sqrt{\frac{E}{12(1-\nu^2)\rho}} = \left(\frac{Ci}{2\pi}\right)\left(\frac{h}{L^2}\right)\sqrt{\frac{1}{12}}\sqrt{\frac{1}{(1-\nu^2)}}\ (\mathbf{C_V})$$

Eq. 12

Ratioing the natural frequency of the new part to the natural frequency of the original part:

$$\omega_2 / \omega_1 = \omega_2 \times (1/\omega_1) =$$

$$\omega_2 / \omega_1 = \frac{Ci_2}{2\pi} \frac{h_2}{L_2^2} \sqrt{\frac{1}{12}} \sqrt{\frac{1}{(1-\nu_2^2)}} \; (\mathbf{C_{V\,2}})$$

$$\times \frac{1}{\frac{Ci_1}{2\pi} \frac{h_1}{L_1^2} \sqrt{\frac{1}{12}} \sqrt{\frac{1}{(1-\nu_1^2)}} \; (\mathbf{C_{V\,1}})}$$

Eq. 13

After inverting the big term on the right side of the multiplication sign (x), gathering like terms, and substituting $h_2 = \alpha h_1$ and $L_2 = \alpha L_1$, the ratio of equations looks like:

$$\omega_2 / \omega_1 = \left[(C_{i\,2} / 2\pi) \times (2\pi / C_{i1}) \right] \times \left[(\alpha h_1 / \alpha^2 L_1^2) \times (L_1^2 / h_1) \right] \times \left[C_{V2} / C_{V1} \right] \times$$

$$\left[(1/12) \times (12/1) \right]^{1/2} \left[(1-\nu_1^2) / (1-\nu_2^2) \right]^{1/2}$$

Note: Eq. 14

1st term: $C_{i\,2}$ and C_{i1} refer to the same mode, so since they are actual numbers, they cancel out. So the first term in Equation 14 cancels out, and the term equals 1.

2nd term: h_1 cancels h_1 in $\alpha h_1 / h_1$ leaving α in the numerator / $\alpha^2 L_1^2 h_1$ / $\alpha^2 L_1^2 h_1$ / $\alpha^2 L_1^2$

L_1^2 cancels L_1^2 in $L_1^2 / \alpha^2 L_1^2$ leaving α^2 in the denomenator, so this term equals α / α^2, or $1/\alpha$.

3rd term: Remains as written.

4th term: Becomes $[(1-\nu_1^2) / (1-\nu_2^2)]^{1/2}$ while (1/12) (12/1) cancels to 1.

And the new equation, solving for ω_2 is

$$\omega_2 = \omega_1 (1/\alpha)(C_{V2} / C_{V1})[(1-\nu_1^2) / (1-\nu_2^2)]^{1/2} \quad \text{Eq. 15}$$

Written more boldly, it is the same equation at which we arrived in the 1st Exercise; except Poisson's ratio term appears, which can "distort" the "protoype" [2], but can also allow fine tuning the frequency prediction if one predicts, say plastic (0.4) from metal (0.3): See Ref. [2], page 641:

$$\boldsymbol{\omega_{n\,2\,i} = (\omega_{n\,1\,i} / \alpha)(C_{V\,2} / C_{V\,1})((1-\nu_1^2) / (1-\nu_2^2))^{1/2}} \quad \text{Eq. 1a}$$

If the materials are the same type, then $\nu_1 = \nu_2$, and "Equation 1" reduces to:

$$\boldsymbol{\omega_{n\,2\,i} = (\omega_{n\,1\,i}\ / \alpha)\ (C_{V\,2} / C_{V\,1})} \qquad \text{Eq. 1}$$

And predicting the new natural frequencies $\boldsymbol{\omega_n}$, or f_n, is as simple as "ratioing" the "velocity of sound" in the two materials.

5.1 ANSWER TO THE "TWO BELL" QUESTION

The "available" mode shapes of vibration in the two identical bells will be exactly the same because the geometry of the bells is exactly the same, but the two geometrically identical bells ring at different frequencies because the "velocity of sound" in the two metals are different. The "velocity of sound" in brass is 11,632 ft/sec, and in titanium, it is 16,625 ft/sec, so the impulse wave in the titanium bell will travel faster than the impulse wave in the brass bell. The higher frequency, shorter wavelength mode shape of vibration in the titanium bell will geometrically fit (respond to) the higher frequency, shorter wave-length excitation wave, and the slower wave in the brass bell will excite (fit) the lower frequency modes in the brass bell. The titanium bell should ring at higher frequencies than the brass bell does, but the "available" modes are the same in the two bells.

6.0 EXAMPLE 3 : PLASTIC (ν = 0.4 to 0.45) → METAL (ν = 0.3) & TWO TYPES OF PLASTICS

In Derivation 2, the fundamental equation for the lateral natural frequencies of a horizontal cantilever plate was rearranged and simplified ("dissected"), and we found that at its "heart" was "**the velocity of sound in the material of construction.**" The "velocity of sound in the material," in conjunction with the Poisson's ratio for the material, the "mode shape of vibration" and the input energy frequency range and input location all define the frequencies and the amplitudes at which that object will vibrate.

Before, the Poisson's ratio had been considered the same for the original and the new materials. The complete form of "Equation 1," which was just derived, says, in effect, as far as predicting the new natural frequencies for a new material which has a different Poisson's ratio, there should be no difference in accuracy in the frequency prediction than there would be in predicting the natural frequencies when both materials are metals, where Poisson's ratios are the same. The equation allows for the Poisson difference in the "$(1 - \nu^2)^{1/2}$ "-ratio correction.

Fig.11 A surprising result from a standard tensile test on a "foamed-core HDPE" sample at 0.2 inches/minute. The plastic sample melted! That is not metal!

with a nylon gear allowed the gears to not have to be lubricated (reducing film contamination), but the "backlash" of the gears had to be adjusted to account for swelling of the nylon gear when it absorbed moisture from the air, and the steel gears had to be hard surfaced because the nylon gears would pick up tiny bits of metal over time and the nylon gear would then machine the steel gear teeth to "nubs."

Figure 11 shows a surprising result which occurred during a standard tensile test on a "foamed-core HDPE" sample at 0.2 inches/ minute. The plastic sample melted! That was not expected. At slower and faster strain rates, this did not occur. At slower rates, the heat generated by distorting the molecules in the material have time to dissipate. At faster strain rates, the material breaks before the heat can build up enough to melt it. At this exact strain rate with this material, in tests on four foamed-HDPE samples, all four samples melted (Appendix D). One does not see that in tensile tests on metals, because metal conducts the heat away.

A concern the author has is that "Equation 1a" looks so simple, engineers working with materials with Poisson's ratio of 0.4 to 0.45, and in particular "plastics," might not have an appreciation for just how much different these materials are. So, in this Section, two different types of "engineered plastics" are analyzed – a bi-axial plastic and a plastic that absorbs moisture.

This Example has three parts:

6.1 The Elastic modulus of a bi-axial plastic (a polyethylene-terephthalate filled with long glass fibers) will be calculated from data provided by the manufacturer and compared with tests conducted data obtained on samples of that plastic,
6.2 Those data will be used to predict natural frequencies for a mild steel part from a plastic part, and
6.3 Finally, the natural frequencies of a blade made of glass-filled, 6,6 polyarimid (nylon) material, both DAM ("dry as molded") and "humidity equilibrated" will be analyzed.

6.1 THE ELASTIC MODULUS OF A BI-AXIAL PLASTIC – A PLASTIC MATERIAL WITH LONG, ORIENTED GLASS FIBERS –

Whenever our engineering group was asked to help solve problems with a design, and particularly when we had not worked with that plastic before, we would ask for samples from the manufacturer. They usually came in the form of flat molded "plaques" (plates) typically 3 in. by 5 in. by 1/8-in. thick (Figure 12).. (Samples are also available as "plaques" 4 in. by 8 in. by 3/16-in. thick, or small samples in the shape of "dog bones.") The plaques were clamped on the edge of a massive steel table in our lab to make cantilever plates, like the one shown in the sketch of Figure 12.

Once again, Equation 4 defining the natural frequencies of a cantilever plate, is re-arranged. It is a very good idea any time you are working with something new, whether it is a new plastic material, a new measurement sensor or a new measurement technology, to try it first on something to which you know the correct answer. A cantilever plate has always worked well. When you can make a measurement on a plastic cantilever plate in the lab and get the correct answer, then you are ready to work with plastics in the field. It is an absolute must that you practice in the lab first.

$$\omega_i = \frac{C_i\, h}{2\pi L^2} \sqrt{\frac{E}{12\,(1-\mu^2)\,\rho}}, \quad \text{Hz} \qquad \text{Eq. 4}$$

becomes

$$E = \frac{4\,\pi^2\,\omega_i^2\,L^4}{C_i^2} \times \frac{12\,(1-\mu^2)\,\rho}{h^2} \quad [\text{psi}] \qquad \text{Eq. 3}$$

Using C_1 from Table 1, the "machine-direction" modulus can then be calculated. The 1st twisting mode, with C_2 taken from Figure 4, can be used to find the "cross-direction" modulus. The C_i coefficients depend on vibration mode of interest and length-to-width ratio, and can be selected from the Table 1, on page 10 of this paper. Note the addition of "μ," Poisson's Ratio. μ = 0.3 for metal and 0.4 for plastics..

The manufacturer's literature reports the "flexural modulus at 73˚ F" of this plastic is 2,000,000 psi. Is it?

Sample cantilever beams were molded from this plastic, and were tested with a strain gage attached as shown in Figure 12. The strain signal goes to a Fourier frequency analyzer, which is set up with "peak hold" averaging. "Plucking" the corner of the plastic cantilever excites both bending and twisting modes of free vibration and the analyzer identifies the first four modes predicted, in this case

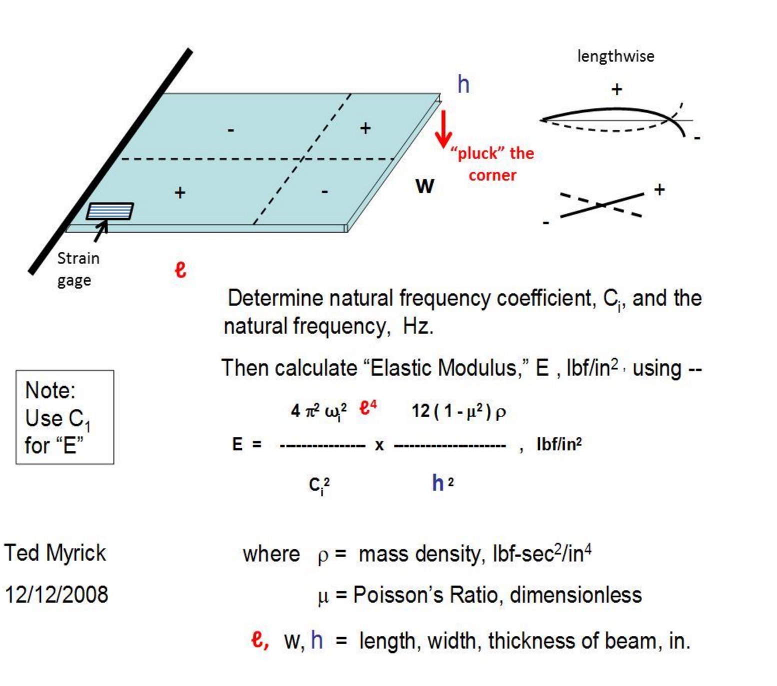


Figure 12. Test set up for testing properties of plastic.

The tests measure "elastic modulus" of the plastic. The beam dimensions are:

Length = 6-1/16 in. = 6.0625 in
Width = 3.976 in.
Thickness = 0.1185 in.

Specific gravity of the plastic = 1.69 , so 1.69 x 62.4266 lb/ft^3 (sp. wt. H_2O)

Specific weight of the plastic = 105.5 lbf/ft^3 = 0.0611 lbf/in^3

Then, the density, ρ, of this plastic = sp. wt. / g_c = 0.0611 lbf / in^3 / 386 in/sec^2 Eq. 12
or

$$\rho = 1.582 \text{ E-4 lbf-sec}^2/\text{in}^4$$

Length / width ratio = 6 in / 4 in = 1.50, and Poisson's ratio = 0.4

Plugging into Equation 4:

ω_i = $(C_i/2\pi)[\ (2E6\ psi)(0.1185\ in)^2)/((12(1 - (0.4)^2)(1.582E\text{-}4 lbf\text{-}sec^2/in^4) \times (6.0625\ in)^4)\ ]^{0.5}$

$\omega_i = (C_i/2\pi)[\ 13{,}035.58\ rad/sec^2\ ]^{0.5}$

finally, $\omega_i = C_i \times 18.17$ Hz. Eq. 16

Determining C_i from Table 1, the predicted frequencies using E = 2.0E+06 psi are compared in Table 4 with the measured values for the first four modes. If E_{flex} is really 2,000,000 psi, the frequencies should agree closely. Two do. Two don't.

We find that the predicted and the measured frequencies of two of the four modes are in close agreement. The two modes that agree are the "bending" modes: (a) the fundamental "diving board" mode at 63 Hz, and (b) the classic second (2nd) bending mode at 384 +/- 2 Hz . The long glass fibers are apparently oriented in the direction parallel to the long axis of the beam. In many casting and extrusion processes, this is called the "machine direction"(MD) and perpendicular to MD is "cross direction," or CD.

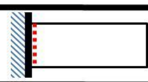
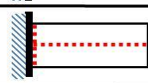
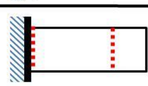
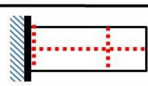
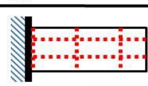
Mode	f_1 measured	E measured	% of 2E6
	62.5 Hz	1.932E6 psi	96.6%
	149	1.017E6	**50.8**
	386	2.047E6	102.3
	565	1.209E6	**60.4**
			

Table 4a

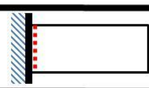
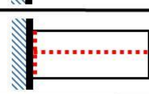
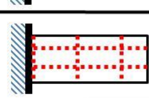
Mode	C_1	f_1 predicted	f_1 measured
	3.5	63.6 Hz	62.5 Hz
	11.5	209	**149**
	21	382	386
	40	727	**565**
	60	1090	

Table 4b

Table 4. Results of testing a beam made from a plastic with oriented long glass fibers ($E_{td} = 2 \times E_{cd}$) and L/W ratio = 1.525. The "elastic modulus" is calculated for the first four modes using the measured natural frequenciesand the Ci for those four modes.

The two modes that do not have agreement between the calculated and measured mode are the modes with free-end twisting about the long axis of the test beam. Apparently, with few of the long glass fibers oriented perpendicular to the long axis, the "cross direction" will be weaker than the "machine direction." This is seen in Table 4A, where one sees the elastic modulus in the "cross direction" is only 50% to 60% of the elastic modulus in the "machine direction."

To do these calculations, the measured natural frequencies are substituted into Eq. 3, and E is calculated. Example:

$$E_1 = [((2\pi \text{ rad/rev})^2) (62.5 \text{ cycles/sec})^2 (6.0625 \text{ in})^4) / (3.5)^2] \quad x$$

$$[12 (1 - (0.4)^2)(1.582E\text{-}4 \text{ lbf-sec}^2/\text{in}^4)/(0.1185 \text{ in})^2] \qquad \text{lbf/in}^2$$

E_1 = 1.932E6 psi , which is 96.6% of the reported modulus. That works!

This is an excellent example of the data from a cantilever beam and "pluck" test on a sample of a new material can generate in just a few minutes time. Damping can also be determined by the "half-power" method.

6.2 PREDICTING STEEL FROM PLASTIC OF SECT 6.1 USING "EQUATION 1a"

For a plastic cantilever beam, or cutting blade on a rotating wheel, $\nu = 0.4$. What would the natural frequencies of that be if we made this small "blade" out of mild carbon steel, which has $\nu = 0.3$?

Cv-md Plastic-R = [2,000,000 x 386 / 0.0611]^0.5 = 9,367 ft/sec and ν = 0.4 Eq. 2b

Cv steel = [29,500,000 x 386 / 0.283]^0.5 = 16,716 ft/sec and ν = 0.3 Eq. 2b

If we employ Equation 1, and ignore ν –

f_n 1 steel = (f_n 1 Plastic-R / α) (Cv steel / Cv rynite) = (63.2 Hz / 1) (16,716 ft/sec/ 9367 ft/sec) Eq. 1

f_n steel = **112.8 Hz** is predicted, ignoring Poisson's ratio.

If we employ Equation 1a , and include ν – 0.4 for Plastic-R and 0.3 for steel

f_n 1 steel = (f_n 1 rynite / α) (Cv steel / Cv rynite) ((1- ν2 / 1- ν2))^0.5 Eq. 1a

= (63.2 Hz / 1) (16,716 ft/sec/ 9367 ft/sec)((1 – 0.16) / (1 – 0.9))^0.5

= 63.2 (1.78456)(0.84 / 0.91)^0.5 = 63.2 (1.78456)(0.9608)

= 108.35 Hz is <u>predicted including Poisson</u>

The actual value of the blade of steel was –

fn 1 steel = 108.4 Hz. <u>Calculated</u> with E = 29.5E+06 psi and ν = 0.3, and Equation 4

and

fn 1 steel = 107.2 Hz Using the **<u>measured</u>** value and "Equation 1a."

That is excellent agreement!

6.3: A PLASTIC WITH SHORT, RANDOMLY-ORIENTED FIBERS --

This is a different plastic material. It is a nylon base material with short mineral fibers randomly oriented in it (except at the surface where the flow of the plastic in the mold tends to orient the glass fibers in the flow direction). Data are below.

Beam length, width and thickness are 5.970 in, 3.945 in, and 0.249 in, respectively. Specific gravity of the material is 1.45, so sp. wt. is 0.0524 lbf/in^3.

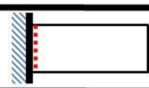
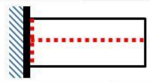
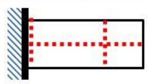
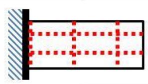
Mode	C_1	f_1 predicted	f_1 measured	
	3.5	105.2 Hz	87 Hz	83%
	11.5	346	292	84%
	21	631	548	87%
	40	1202	997	83%
	60	1804	1395	77%

Table 5. Results of testing a beam made from a plastic with random, short mineral fibers ($E_{td} = E_{cd}$).

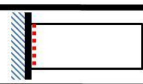
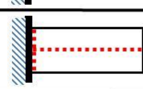
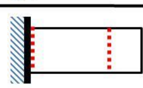
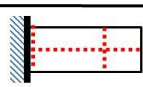
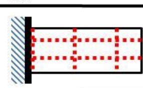
Mode	f_1 measured	E measured	% of DAM
	87 Hz	683,600 psi	68.4%
	292	713,300	71.35
	548	753,400	75.34
	997	687,340	68.7
			

Table 6. Elastic modulus for a mineral-glass filled, nylon-like material which has been exposed to moist air. The DAM elastic modulus is reportedly 1,000,000 psi, and 600,000 for the 50% RH equilibrated material. This material average of 71% of DAM (“dry as molded” and immediately sealed) elastic modulus.

The predicted values in Table 6 are based on the “dry as molded” (DAM) modulus. This material absorbs moisture and loses strength, however, so the measured values are consistently lower than the predicted values. Note that the material is the same in the “md” (machine direction) and “cd” (cross direction) directions. That is, “cd” is perpendicular to “md”.

These tests are quick, using the equation for "E" in Figure 12, and accurate, as long as the natural frequencies are accurately determined. Make sure no heavy sensors are used and the base of the cantilever is securely clamped to a heavy, rigid test table. The tests can be done and the report written in just 4 hours, typically.

"Equation 1a" also holds true for some non-metallic materials. We tested it on steel-reinforced concrete versus natural frequencies predicted from measurements on an aluminum plate. As shown in Section 8.0 of this paper, the predictions were very close to those found in the concrete foundation table. For other materials (see Table 7), such as the hard, dense "paperboard" cylindrical tubes on which nylon and Dacron fiber is wound during the spinning process, compare the natural frequencies predicted for them from measurements on a steel or aluminum cylinder. A test like that just described is recommended when any material other than normal materials of construction are used.

Material	C_v - Velocity of Sound (ft/sec)	E = Modulus of Elasticity (psi)	γ = Specific Weight (lb/in^3)
Aluminum	**16,375**	**10E+06**	**0.100**
Brass (70-30)	**11,632**	**15.4E+06**	**0.305**
Concrete	**11,823**	**4.7E+06**	**0.0897 (155 lbf/ft3)**
Copper	**11,670**	**16.3E+06**	**0.321**
Iron	**16,410**	**28.4E+06**	**0.281**
Paperboard Tubes	**12,945**	**1.574E+06**	**0.0252**
Steel	**16,715**	**29.5E+06**	**0.283**
Air, dry, 32°F	**1,088.5**	**N/A**	**4.676E-05**
Water	**4,794**	**N/A**	**0.0361**

Table 7 – Velocity of Sound, Elastic Modulus, and Specific Weight
$C_v = (E/\rho)^{1/2}$; $\rho = \gamma/g_c$

How did Equation 1 work in real problems in plants? Here are two examples.

7.0 EXAMPLE #4: SIEVE TRAY CRACKING PROBLEM (METAL → METAL)

"Equation 1" Accurately Modeled The Natural Frequencies Of A Sieve Tray In A Distillation Column, Which Had A Cracking Problem

One of our plants had a severe cracking problem with "integral beam" panels (Figure 13) in a 12-foot diameter dehydration column, but there were no spare trays to test to compare possible solutions. The plant wanted us to compare various "clip" and "strap," and even "bolt-loosening," proposals to see which one reduced dynamic stress in the trays the most. We concluded, at that time, we would be more confident in a physical structure to test with real "boundary conditions," than a computer model with someone's "estimated" boundary conditions.

The trays were perfect for a "true scale" model test in that:

1. All dimensions could be (and were) changed by the same "scale factor," "α;"

2. It was all one material (except for the steel bolts in the model)

3. Trying to model nearly a dozen thin panels connected at discrete points by a couple hundred bolts would have been a problem for finite-element computer users.

And how would someone model "hand tight" bolts with a computer model? With the scale model, one just decides which bolts to loosen -- and loosens them. Or slowly tightens them during the test to see at what torque value the bolts start to get the seams tight enough to cause the panels to vibrate as a continuous plate. The next tests can be underway immediately. This was particularly true since the trays had hundreds of bolted connections. We could try several possible "fixes" per day, but it would take "hours and hours" to change the computer each time someone came up with an idea.

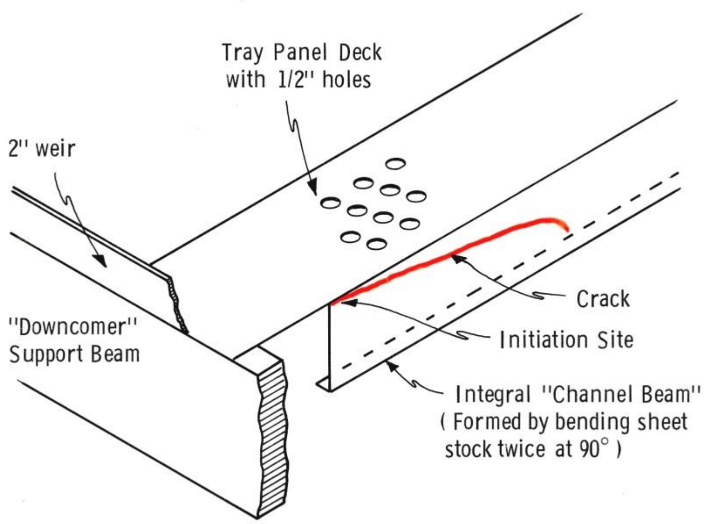


Figure 13. The "integral-beam" tray is made of several panels bolted together. .

Each panel of the tray shown in Figure 13 has a horizontal section with lots of holes and another part with no holes. This part with no holes is folded first to form the vertical "integral beam" and then a narrow section was bent to form a flange on the bottom, which also helps to stiffen the panel. The drawback to this design is a sharp corner where the cracking started. Eventually the cracks propagate and then pieces of the trays break off and fall down on the trays below, After a few tray failures, the column doesn't work.

7.1 SCALE FACTOR

So, what decision do you have to make first when designing this model? If you change the size, all dimensions have to be changed by "α." The thickness of aluminum sheet stock was the critical dimension. We had on-hand 12-gage aluminum sheet stock which was approximately one-half the thickness of the 16-gage 316-L stainless steel trays. This meant our model would be approximately 6-feet in diameter, or (α = ½). See Figure 14. That turned out to be a good size. It was big enough to look like and act like a real tray, but small enough that one person could easily handle the parts. The calculated (predicted) natural frequencies of the aluminum scale model were **2.11** times higher than the frequencies measured on an actual tray at the plant. This scale factor is calculated to be:

$$\omega_{imodel} = (\omega_{iplant} / \alpha)(C_{vm} / C_{vp}) \qquad \text{Eq. 1}$$

$$= (\omega_{ip} / (1/2.078)(C_{vn} / C_{vo})$$

C_{vm} (aluminum) = 16,375 ft/sec

C_{vp} (stainless steel) = 16,100 ft/sec

so,

$$\omega_{imodel} = \omega_{iplant} (2.078)(16{,}375/16{,}100)$$

$$= 2.11 \ \omega_{iplant}$$

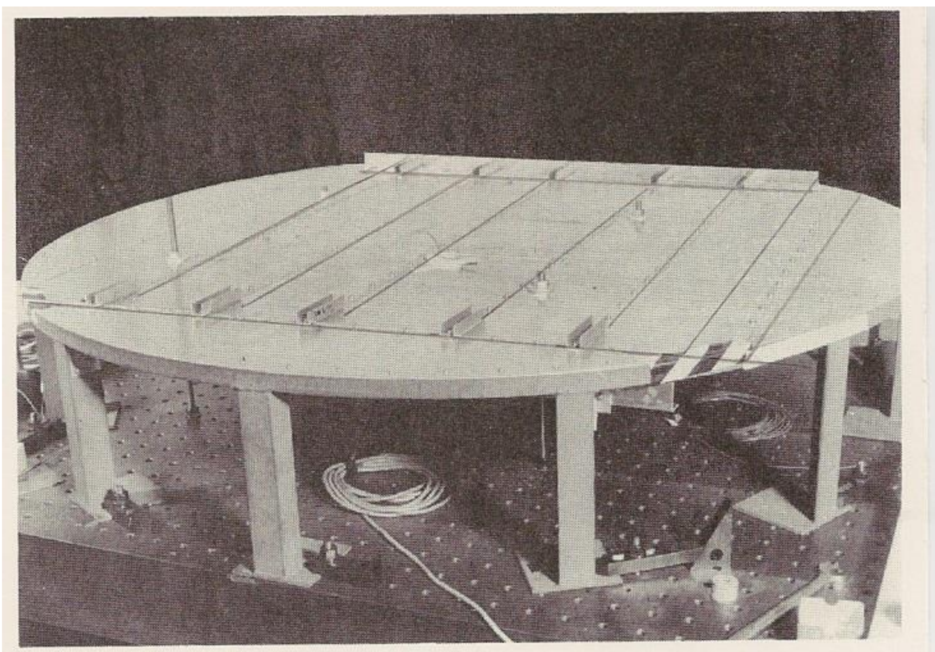


Figure 14. A half-size (6-ft diameter) aluminum sieve tray model. The model tray is bolted to a steel ring just as the 12-foot tray is in the column

The aluminum model tray was mounted on a steel ring on a steel test table (Figure 14) in one of our labs. It took a week to make. Since it was aluminum, it was easy to handle and modify. We could try everybody's favorite idea to reduce the stresses in the tray. Some of the ideas helped in some parts of each tray, but not in others, and some actually would have increased the corner stresses! We could try two or three modification proposals in a day. A computer model could not do that. And how would we have modeled hundreds of bolted connections – some tight and some loose – on the computer?

The tens of dozens of process holes in the actual tray were not drilled in the horizontal sections of the model, because they weren't needed. The mode shapes of the 6-foot model tray matched the measured modes shapes of the actual 12-foot tray without the time and cost required to drill all those holes in the model.

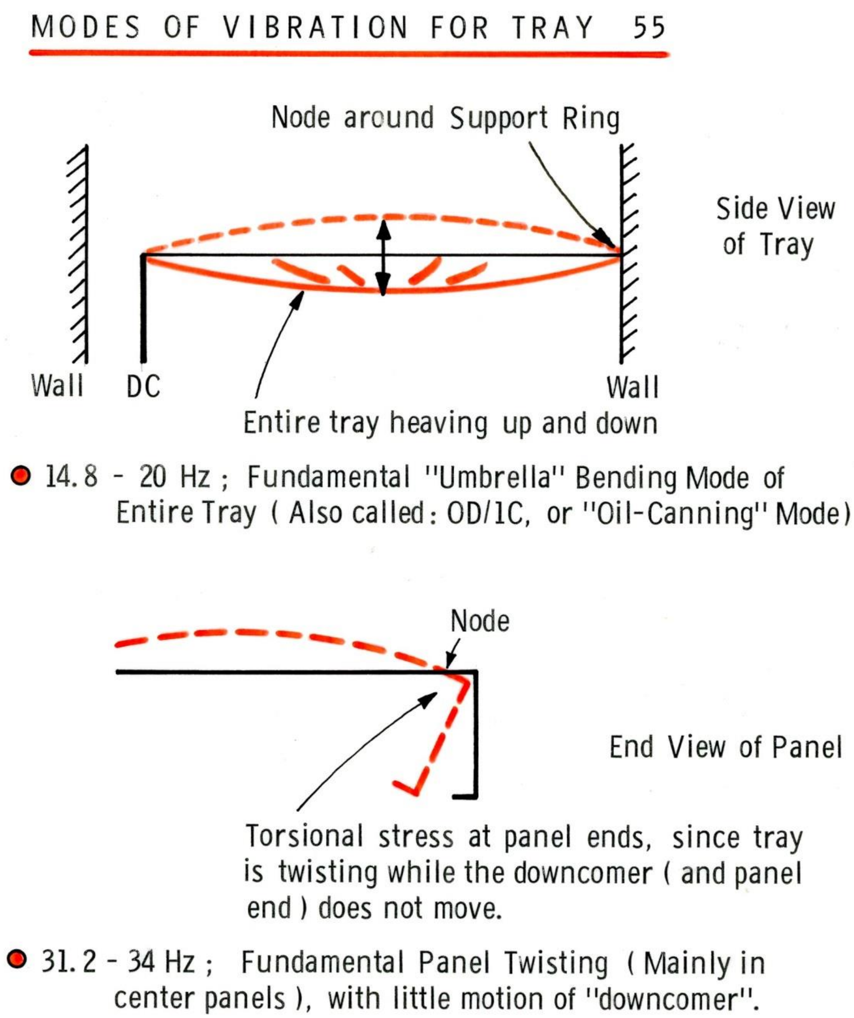


Figure 15 Two modes of vibration of the trays in the 12'-dia. column. The 31.2 – 34 Hz mode is a "standing wave" with alternating panels moving up and down in a sinusoidal pattern.

Two modes of vibration of "tray 55" measured with the 12-ft-dia. tray installed in the column are shown in Figure 15. The 14.8 to 20 Hz mode is the fundamental vertical "drumhead" mode. It is also called the "trampoline," or "oil-canning," mode. The 31.2 – 34 Hz mode created a "standing wave," with panels moving up and down in a sinusoidal pattern out-of-phase with the panels on either side.

Table 8 shows how accurately "Equation 1" predicted the natural frequencies in the aluminum model using the measurements of natural frequencies in the column. With the model to test, we could identify frequencies in the operating data which we did not find in the static tests. We took the frequency found in the operating data, multiplied it by a factor, $\alpha = 2.11$, and then tuned the shaker to approximately that frequency. The frequency was then shifted up and down a little until the localized resonance in a portion of the tray showed itself.

The purpose of this paper is the prediction of natural frequencies, not how the problems were addressed, so the rest of the story will just be summarized for completeness. The continuous monitoring of "tray 55" in the operating column by the accelerometer in the stainless steel protective cup showed vibration consistent with tray failures caused by "standing waves." That mode is shown in Figure 15 at 31.2 to 34 Hz.

NATURAL FREQUENCIES OF THE ACTUAL TRAY (# 55) MEASURED IN THE COLUMN COMPARED WITH NATURAL FREQUENCIES PREDICTED AND MEASURED FOR THE 1/2-SIZE MODEL IN THE LAB

Actual Tray # 55 Measured in Column At The Plant		Model Tray (α = 1/2) Predicted From The Full-Sized Tray fn's		Model Tray (α = 1/2) Measured On The Model Tray In The Lab	
20	Hz - 1st mode	41.5	Hz	39 - 45	Hz
24 - 25	"	50 - 52	"	50 - 53	"
28 - 30	"	58 - 62.5	"	56 - 60	"
34 - 35	" - Twisting	70.5 - 73	"	67 - 73	"
40 - 42	"	83 - 87	"	(82 - 87)	"
48 - 49	"	100 - 102	"	(92 - 102)	"
51 - 51.8	"	106 - 108	"	(105 - 106)	"

Table 8. Very accurate frequency predictions. We had confidence that vibration reported at a particular frequency meant a certain mode of vibration was being excited

The vibration band in the actual column had a center frequency which changed as the "total vapor rate" of the column changed. It looked like the response of a mechanical structure during a vibration test with the excitation from an electromagnetic shaker being controlled by a "sweep oscillator."

Three-axis rosette strain gages were attached to the model to measure maximum alternating stresses in the sharp corners where cracks started in the real trays (Figure 13) and a large electromagnetic shaker (Figure 16) simulated the excitation caused by gas flowing up the column while liquid spilled down from the top.

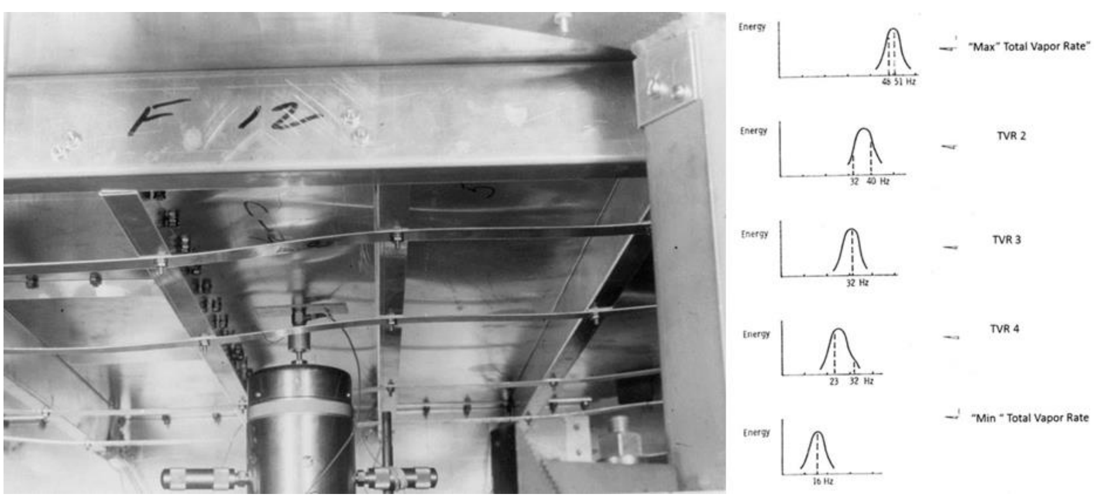


Figure 16. Four of the panels of the 6'model showing the underside of the tray and the shaker which simulated the narrowband response seen with the accelerometer in the operating column. Frequency was function of the "vapor rate" of the column.

7.2 FIELD TESTING: MODAL ANALYSIS ON TRAY "55" IN THE COLUMN

We used "Equation 1" to design the ½-size model of the bolted-construction sieve tray. It was necessary that mode shapes and vibration frequencies be determined for the full-size tray in operation to see how close were the predictions. This required getting a person (s) and all necessary equipment up to the correct elevation, working partly inside the column, but not so much that the person would affect the results, and squeezed between trays. This was very difficult since "Tray 55" was over 50 feet up from ground level, 12 feet in diameter, and separated vertically from trays above and below it by only 12 to18 unches.

I owe much to my good friend, "Ed," a PhD engineer who is an expert on modal analysis. Ed agreed to the job without knowing he would have to work off a platform hanging from a crane over 50' up on the column. And he did not know he suffered from "acrophobia" (fear of heights) and claustrophobia (fear of small spaces) either, until he did this work. Somehow, he did it well. Figure 15 and Table 8 are testament to that conclusion. Without "Ed's" efforts, this wouldn't be much of a story. Thank you, Ed.

7.3 WHAT WOULD A COMPUTER MODEL HAVE ADDED?

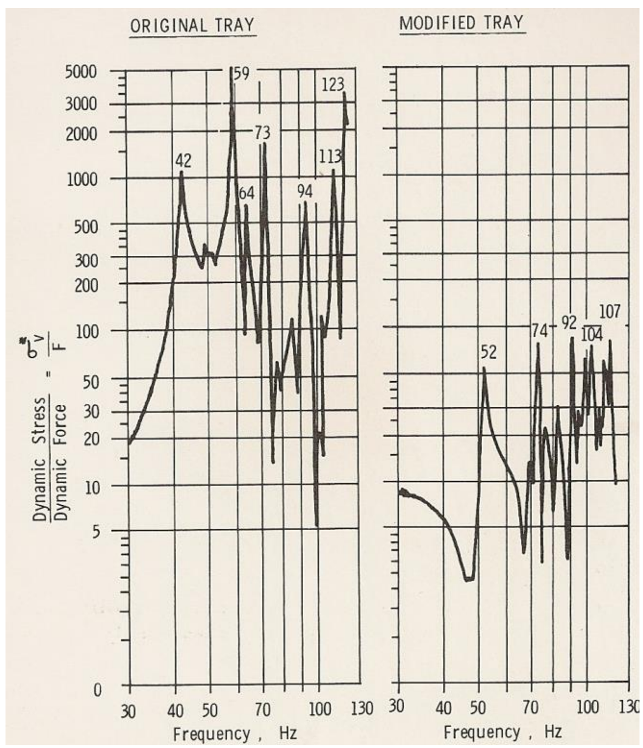


Figure 17. Comparing resonant dynamic stress per unit dynamic force for the strain gage located next to a sharp corner in an unmodified tray (on the left) versus that same gage (on the right) after the tray was modified with "down-comer clips."

What we lost without having a computer model was the distribution of stresses. We had stresses at particular points where our strain gages were located. The gages kept track of the stress at that point, but did not tell us where possibly higher stresses were, or how sharply some of the gradients changed. The best approach would have been to have both the computer model to look at stress distribution, and the measurements to verify the stresses at gage locations. We could then estimate the dynamic stresses at locations between strain gages. The stresses shown in Figure 17 show a tremendous reduction in stress at the sharp corner of Figure 13, but as it was pointed out in one of our reviews, all that energy in the column has to go somewhere. The stress at the corner of Figure 13 is now small, but it might be worse than it ever was at some other location.

7.4 DYNAMIC STRESSES WERE NOT MEASURED ON THE NEW TRAYS

"Dynamic stresses in a corner without and with the "clip" (the repair method) on the model are shown in Figure 17. Of course, these are stresses on the model tray with and without various modifications while being shaken in a laboratory. Actual operating dynamic stresses in "Tray 55" after the clips were installed were never obtained. Somehow the plant started up before the accelerometer could be reinstalled on the tray. The modifications took longer than we thought, startup pressure was applied by Operations, and the fact that they were going to shut down and put the new trays in anyway meant no one except a couple of research engineers were interested in how well the predicted stresses matched the actual stresses.

The equation for static stress generated by liquid weight and vapor pressure across the tray was:

$$\pi_4 = M c / (\sigma I) = \text{dim} = \pi_{4'} \qquad \text{Eq. 17}$$

$$\sigma_a = \sigma_m / \alpha^2 \; (F_a / F_m) \qquad \text{Eq. 18}$$

$$= (\sigma_m / F_m)(F_a / \alpha^2)$$

where Force, F, is the "equivalent force" due to liquid on the tray and vapor ΔP that was applied as a point load. The work was never completed and we offer no guarantees as to the accuracy of any estimated forces or stresses.

7.5 HOW WELL DID IT WORK?

The plant made it, but I heard some of the clips had failed after only a few weeks of operation. The repair approach "worked" since it's goal was to keep the plant running until the "loose beam" trays arrived, but we heard the construction mechanics who had been tasked to put the hardware in on an earlier shutdown were not happy with us because of all the hundreds of "clips" they had to install, using hand drills, bolts and wrenches with trays and "down-comers" separated by roughly a foot!

The cost of field measurements and laboratory tests, plus the cost of the model tray, in the late '70s, was $35,000. The entire effort took about one month with one engineer and two lab technologists, plus others at times, doing the work.

8.0 TECHNOLOGY EXAMPLE #5 –

PREDICTING THE NATURAL FREQUENCIES OF AN 880-TON STEEL-REINFORCED CONCRETE, "LOW-TUNED" ("SPRING ISOLATED") FOUNDATION TABLE

BACKGROUND AND SUMMARY:

In the late 1970's, "physical-model" technology was used to predict the natural frequencies and associated mode shapes of vibration of a 880-ton spring-supported concrete foundation "table" (Figure 19). The top surface of the table was 35-feet above ground. The big slab of steel-reinforced concrete was supported by sixteen (16) "springs" positioned atop ten (10) concrete pedestals. Each of the sixteen springs supporting the table were "a stack of Belleville washers in a can" (Figure 19). The full-size table was vibration tested for mode shapes and frequencies before machinery was installed, and then again after 175 tonnes of machinery, piping and equipment were installed. This modeling was done for two reasons: 1). We were exploring other ways of analyzing new equipment purchases to ensure smooth, trouble-free plant startups, and 2). we had no foundations of this design.

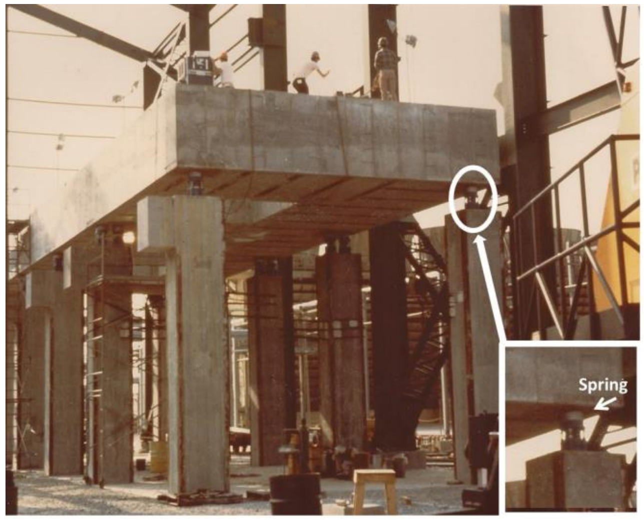


Figure 18. An 880-ton concrete, spring-isolated compressor platform.

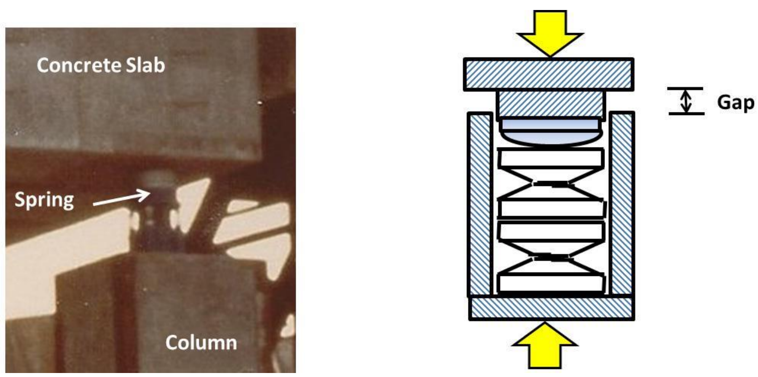


Figure 19. A sketch of what a spring assembly might look like inside.

8.1 A "LOW-TUNED" ("SPRING ISOLATED") FOUNDATION

The concrete slab is a support platform (or "table") for a large motor-driven compressor train. A "low-tuned" foundation means the heavy platform is supported on springs sized such that the platform with machinery mounted on it has a fundamental vertical, rigid-body 1st natural frequency (the 'bouncing" mode) between 1/4th and 1/3rd of the lowest major excitation. In this case, the motor operates at 1200 rpm and the 1st vertical bounce mode is designed to be less than 1/3rd of that frequency.

Referring to Figure 21, typically 90% to 95%, or even more, of the unbalance forces are "isolated" at the machine level; that is, they are not transmitted to the support system at and below grade (ground level). Spring-isolated, or "low-tuned," foundations can be used in marshy areas, or otherwise soft soil conditions, to prevent machine unbalance forces from continuing to drive the support piles. If the support piles continue to sink, the platform could settle or tilt.

Using spring stiffness values provided by the design firm, our model predicted the “1/1” plate mode would be right on 20 Hz (or 1200 rpm), which is coincident with motor running speed. The centerline of the 7,000 rpm compressor train would be almost on the node line for the 1/1 mode, so the compressors would be fine; however, people standing on the corners of the table would feel “moderate to severe” vibration. Following our recommendation, the frequency of the 1/1 mode was changed in a few hours by restacking the “Belleville” washers in the four corner spring cans. This allowed both the compressor train and the operators standing beside the machine to be “comfortable.”

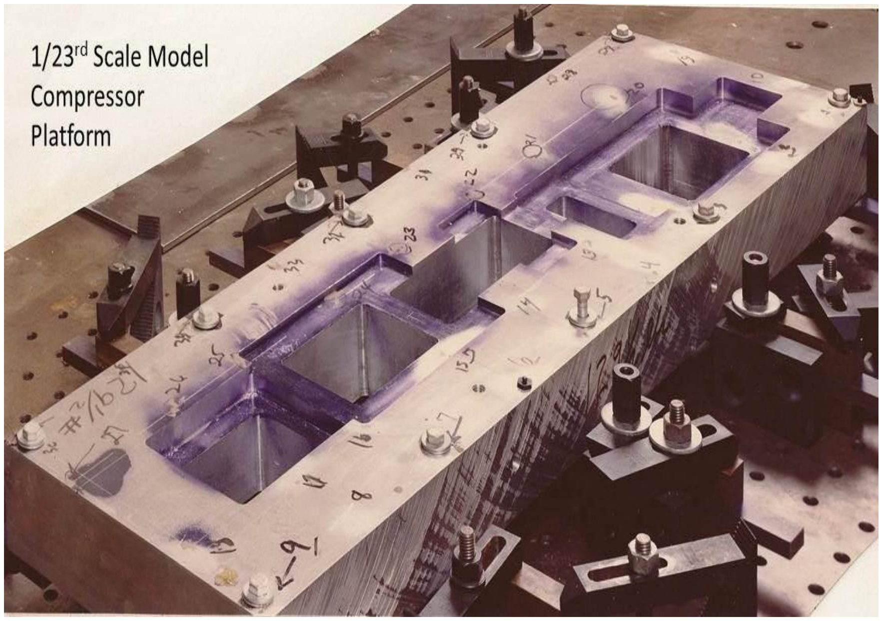


Figure 20. The 1/23rd aluminum “table” model supported on little cantilever beams.

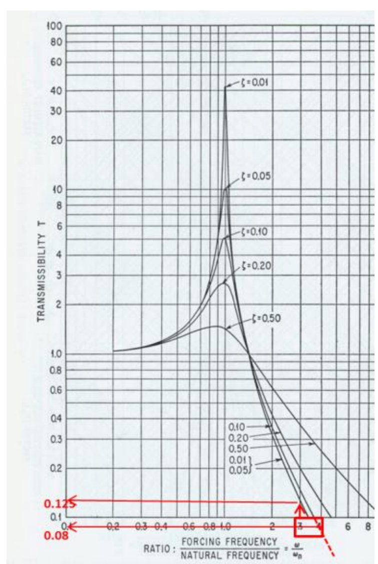


Figure 21. A “low-tuned” platform would have a speed ratio of 3.0 to 4.0,
and an isolation effectiveness of 88 to 92%.

Step 1: Identify the lowest excitation frequency of the equipment on the platform. In this case, it is the 1200 rpm (20 Hz) synchronous motor driving the compressor train. The compressors run at 7000 rpm and are not even an issue. Higher frequencies are more easily isolated.

Step 2: Determine the required natural frequency of the platform. $1/4^{th}$ to $1/3^{rd}$ of 20 Hz is 5 Hz to 6.67 Hz. The fundamental up-and-down "bounce" mode natural frequency of the concrete platform has to be in that frequency range.

Step 3: Determine the total weight of the platform with the machinery on it, including piping. In this case, it is 800 tonnes. Converting units -- 800 tonnes times 2,205 lbf/ tonne = 1,764,000 lbf ; that is, the concrete, machinery, and piping weight.

Step 4: Determine the "deflection" (Δ) of the springs which will give the desired frequency carrying this load. Refer to page 5, equation 1.2-10, and Fig. 1.2-2, in William Thomson's excellent textbook, VIBRATION THEORY AND APPLICATIONS (Ref. 7). This problem is the first problem students tackle in a vibration course – the natural frequency of a mass sitting on an undamped spring. The equation is:

$$f_n = (1 / 2\pi) \sqrt{k / m} \text{ , Hz.} \qquad \text{Eq. 19}$$

This can be rearranged to

$$f_n = 3.127 / \sqrt{\Delta} \text{ , Hz.} \qquad \text{Eq. 20}$$

In this expression, f_n is in "c.p.s." or "Hz."

"Δ" is the "deflection" of the spring when a mass is placed on it. The fundamental natural frequency at which that mass on the spring will vibrate is f_n, or the 1^{st} natural frequency. In this case, the 1^{st} natural frequency must be in the range from 5 to 6.67 Hz for this installation per Step 2 above. We'll choose f_n to be 5.45 Hz.

Solving Equation 20 for "deflection," in inches:

$$\Delta = (3.127 / f_n)^2 \qquad \text{Eq. 21}$$

$$= (3.127 / 5.45)^2 = (0.57376)^2$$

$$= 0.3292 \text{ in.}$$

So when the jacks are removed and the 1.764-million pound concrete slab is lowered onto the 16 springs, the springs will deflect downward by an average of 0.3292 inches.

Step 5: Determine the total stiffness, K. The equation for spring force is:

$$F = K\,\Delta \, . \qquad \text{Eq. 22}$$

The force is the spring rate, in lb/inch, times how much the spring is deflected, in inches. Solving for "K", the total stiffness of 16 springs in "parallel," yields:

$$K = F / \Delta = 1{,}764{,}000 \text{ lbf} / 0.3292 \text{ in} = 5{,}358{,}445 \text{ lbf / in.}$$

To support the total weight of the platform and machinery on 16 springs and have the springs deflect 0.3292 inches, the total stiffness of the springs must be 5,358,445 lbf/inch.

Step 6: Determine the stiffness of each spring in the full-size design. There are 16 springs. Each spring, initially, will be stacked the same and have the same spring rate. The stiffness will be 1/16th of the 5,358,445 lb /in total. So, dividing by 16, each spring will be: **335,000 lbf /inch.** That is exactly what the designer of this foundation told us the spring rate would be for the 16 individual springs.

Step 7: Check it:

$$f_n = (1/2\pi)\sqrt{K/M} \qquad \text{Eq. 23}$$

$$= (1/2\pi)\,(16\,(335{,}000\text{ lbf/in}) / (1{,}764{,}000\text{ lb} / 386\text{ in}/\text{sec}^2))^{0.5}$$

$$f_n = 5.45\ \text{Hz}$$

Exactly right. And in Figure 21, the computer prediction is 5.4 Hz..

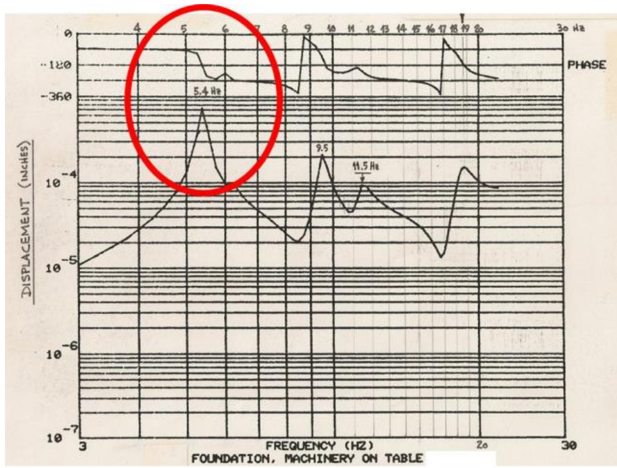


Figure 21. Early computer analysis trying to predict deflection.

The computer analysis trying to predict deflection amplitudes at various frequencies used the maximum allowable motor unbalance as the forcing function and 3% damping. It predicts 0.7 mils at 5.4 Hz (on coast down, maybe?) and 0.10 mils at motor speed of 20 Hz. Vibration amplitudes at the four corners were predicted to be in the "Troublesome" range with the original spring stiffnesses.

The large platform is supported by heavy-duty "Belleville" washer-type springs in a steel can, also called "tapered cone" springs. I do not know the design details of these assemblies; I know only that each cylindrical "spring can" had several Belleville springs inside in a stack. A spring assembly (Figure 19) can be seen supporting the corner of the platform in the upper right corner of Figure 18. The middle three columns on each side have two springs each.

The stiffness of each spring stack can be altered by re-orienting the individual washers in the stack (flipping the springs over), as shown in Appendix F. Belleville springs don't take up a lot of space and are ideal where small deflections are required.

8.2 DETERMINE THE CORRECT STIFFNESS FOR THE MODEL "SPRINGS" USING EQUATION 1.

A 1/23rd size scale model of the foundation table (Figure 20) was machined from one big chunk of aluminum. The springs in the model were cantilever beams. These are not scale models of Belleville springs. A scale model spring would have included tiny Belleville washers in something that looks like a little paint bucket. If it ever looks like some of the problems might be caused by the washers inside the cans, then we would need to go to tiny spring assemblies, but we do not need that amount of detail here.

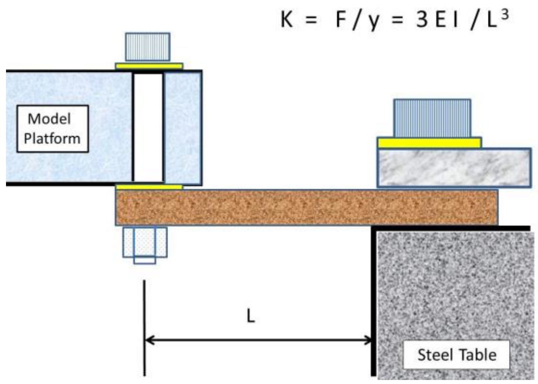


Figure 22. Cantilever beams (brass bars) support the aluminum model.

8.3 ONCE AGAIN – USE "EQUATION 1"

The foundation designer supplied the stiffness, "$k_{effective}$", for each spring assembly. My notes say that the "spring stack" stiffness was **335,000 lbf/inch** (60 tonnes / cm), and the earlier calculations agree. Since the cantilever beams are not models, how does one relate the model support springs to the actual concrete platform on Belleville springs?

If there were to be a "plate" mode coincident with motor running speed, then $f_?$ would be at 20 Hz. Remember "Equation 1" for "true" scale models? The 130-lbf aluminum model is a "true' scale model, and the model, if it is to be accurate, should have mode "$f_?$" and model motor speed coincident, as well.
From "Equation 1"

$$\omega_{ip} = (\omega_{im} / \alpha)(C_{v\text{-}p} / C_{v\text{-}m}) \qquad \text{Eq. 1}$$

where the "prototype," (p), is the full-sized concrete foundation (C_v = 11,823 ft/sec), and the aluminum "model " (m), (C_v = 16,375 ft/sec). Solving for the "i-th" mode of the model:

$$\omega_{im} = (\omega_{ip} \times \alpha)(C_{vm}/C_{vp}) \qquad \text{Eq. 1'}$$

For the "plate" mode in the real foundation at 20 Hz, the model frequency would be:

$$\omega_{im} = \mathbf{20\ Hz \times 23 \times (16{,}375\ ft/sec\ /\ 11{,}823\ ft/sec) = 637.1\ Hz}$$

and motor speed on the model would then be 637.1 Hz.

So, we need then to match the model spring forces to the "i"-th mode. This is done by recognizing that from Figure 4, for "true" scale models, the ratio of natural frequencies remains the same (see discussion in Section 3.9). Therefore, for the full-sized table on springs, **if** there is an "i-th" mode at 20 Hz, the ratio of this mode to the "bounce" mode is 20 / 5.45 Hz = 3.67. The same ratio should hold for the "i-th" mode of the model, so the "bounce" mode must be 637.1 Hz / 3.67 = 173.6 Hz.
Now use equation "Thomson 1.2.-2," and determine the deflection for the "bounce" mode:

$$\Delta = (3.127 / f_n)^2 = (3.127 / 173.6\ \text{Hz})^2 = (0.018)^2 = 3.245\ \text{E-4 in.} \qquad \text{Eq. 24}$$

For the cantilever beam of Figure 22:

$$K = F / X = F / \Delta = 3 E I / L^3 \qquad \text{Eq. 25}$$

For K equivalent to 16 springs, knowing the aluminum plate model weighed 130 lbf:

$$K = 130 \text{ lbf} / 0.0003245 \text{ in} = 400{,}616 \text{ lbf / in}$$

Dividing by 16, each spring on the corner will be **25,039 lbf/inch = k_{model} .**

And each of the three inner spring positions will be twice (2x) that.

For the corner springs, solving for "L" in Figure 22, for a 1" x 1" brass cantilever beam, the area moment of inertia of the brass cantilever should be:

$$I = b h^3 / 12 = (1")(1")^3 / 12 \qquad \text{Eq. 26}$$

$$= 0.0833 \text{ in}^4$$

And using the equation in Figure 22:

$$L^3 = 3 E I / k \qquad \text{Eq. 27}$$

$$= (3 \times 15.4 \text{ E6 lbf /in}^2 \times 0.0833 \text{ in}^4) / 25{,}309 \text{ lbf / in} = 152 \text{ in}^3$$

or

L_{corner} = 5.34 inches

and the beams on the **six inside positions** will be

L_{inside} = 4.24 inches long.

As before, CHECK IT!

$$k = 3 E I / L^3 = 3 \times 15{,}400{,}000 \text{ psi} \times 0.0833 / (5.34)^3 = 25{,}273 \text{ lb/in} \qquad \text{Eq. 28}$$

$$fn = (1/2\pi) (16 \text{ springs} \times 25{,}273 \text{ lb/in/spring} \times 386 \text{ in/sec}^2/130 \text{ lb})^{0.5} \qquad \text{Eq. 29}$$

$= 174.4$ Hz , which is the vertical "bounce" mode frequency.

Close enough. Round-off error is the difference.

And **the ratio to convert a frequency in the model to a frequency in the full-sized concrete platform** is:

$$F_{full\text{-}sized} = (f_{model} / \alpha) (C_{v\ concrete} / C_{v\ aluminum}) = (f_{model} / 23) (11{,}823 \text{ ft/sec} / 16375 \text{ ft / sec})$$

$$F_{full\text{-}sized} = 0.0314\ f_{model} \qquad \text{Eq. 30}$$

And to convert a frequency in the full size platform to the model is:

$$\text{Model} = 31.85 f_{full\text{-}sized} \qquad \text{Eq. 31}$$

Checking one last time:

174 Hz in the model x 0.0314 = 5.46 Hz = fundamental "bounce" mode

637 Hz in the model x 0.0314 = 20 Hz = motor speed. Both good!

To be careful, the model cantilever-beam springs were load-deflection tested to get actual plots of force on the beam, F_m, versus brass beam deflection, y_m. This accounted for flexibility of the fixture on the table, looseness, errors in tolerances, etc.

8.4 TESTING AND RESULTS

A. IN THE PLANT –

A 1000-lbf hydraulic shaker was fastened to steel plates grouted in at various locations on the full-sized platform, and amplitude mode shapes were determined experimentally at the various natural frequencies using the "stationary and roving accelerometer" method. With this procedure, one accelerometer remains at the excitation location (the shaker) as "Reference," and the "Roving" accelerometer finds a node line and follows around the platform. Something like chalk (on concrete) is used to map the mode at each frequency of interest. This is the activity taking place on top of the slab shown in Figure 18 at the start of Section 8.

B. IN THE LABORATORY --

A 1/23rd size scale model of the foundation table (Figure 20) was machined from one big chunk of aluminum. The foundation "table" was supported by 16 cantilever beams in the lab tests with the beams attached to a massive steel table. The aluminum model predicted the natural frequencies and mode shapes of the full-size table more accurately than a computer model did in a parallel study, although both were quite good.

Roughly the same procedure was used with the aluminum model; except, instead of large steel plates embedded in concrete, small steel washers were glued to the aluminum model, and instead of a 1000-lbf hydraulic shaker, a 1-lbf Wilcoxin® electromagnetic shaker was used excite the model. Results are listed in Table 9.

Testing the model in our lab predicted a **"1/1plate mode**" of the concrete platform table at 1150 cpm (see Figure 23). The measurements on the actual foundation showed the 1/1 mode at 1191 cpm. Motor speed is 1200 rpm.

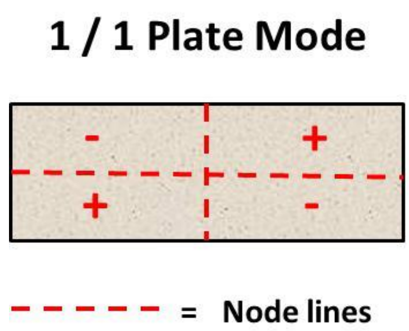


Figure 23. The " 1/1 " plate mode.

The computer animated mode shape is shown in Figure 24. When this mode is excited, the platform is twisting around the lengthwise centerline and also about the transverse centerline. The highest vibration amplitudes are at the corners. When the lower left corner and upper right corner are going up, the other two corners are going down. The part that is going down is called "out of phase" with the part that is going up.

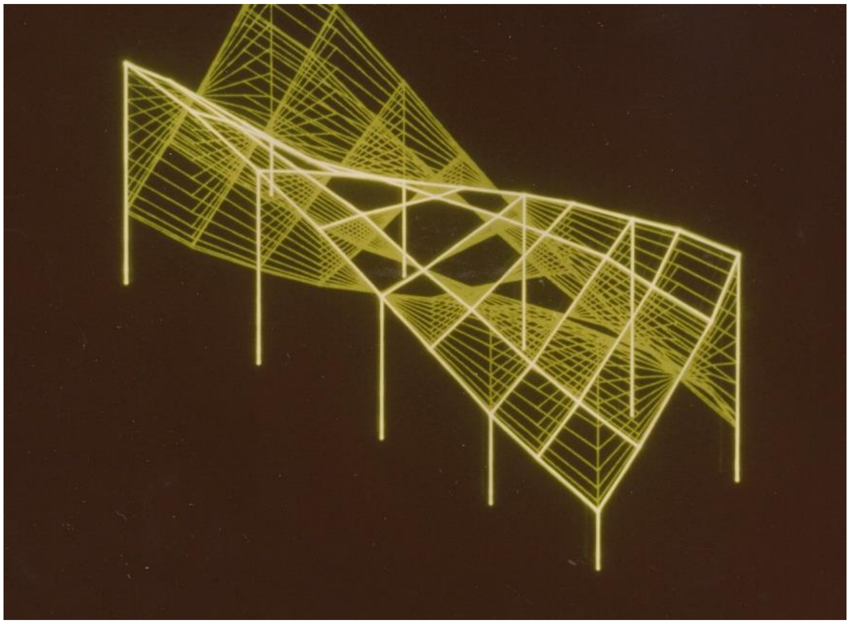


Figure 24. Animated display of the “1/1” mode of vibration of the concrete platform

This does not necessarily mean trouble for the compressor train, since it is positioned very nearly on a lengthwise node line, but it does mean higher vibration of the corners where operators stand.

8.5 Doubled The Stiffness Of The Four Springs On The Corners

I am not an expert on springs made from stacked Belleville washers. Not much to see stuck up under a big concrete slab. The 1,000-lbf hydraulic shaker attached to one of several plates embedded in the “table” when tuned to a natural frequency of the “table” could excite the tall lights on the platform causing them to sway back-and-forth at an amplitude of several inches.

Using the ratio of vibration amplitudes measured at the four corners of the table to amplitudes measured at the bearing locations during the full-size tests gave us an idea what the operating vibration levels could be when the unit went into operation. If vibration at the bearings were in the “high-but-still-acceptable” vibration range at the bearing with the compressor running, vibration at the corners of the table would be too high” when the unit went into operation ... estimates pointed to the “troublesome” range. Our recommendation was to double the stiffness of each corner spring.

The work platforms around the bearings were still in place. Hydraulic jacks were used to raise the concrete table, one corner at a time, just enough to slide the spring unit out; that is, after a steel “safety block,” was put in the place where the spring had been. By rearranging the heavy “Belleville” spring washers in the “spring can” the stiffness of the spring can be changed. The washers were removed, cleaned, inspected, re-lubricated, and put back in the can arranged like the spring in Figure 4. The jacks were put back in place, the platform was lifted a little bit, the stop was removed, and the platform was lowered onto the stiffer spring. After the four springs were changed, we used the hydraulic shaker system and confirmed the frequencies went where they were supposed to go. It reminded me of changing a tire, but it took longer—about half a day for four springs.

Vibration Mode	Natural Frequencies (Hz)		
	Measured on the Actual Concrete Table	Predicted by the Scale Model	Predicted by the Computer Program
Lengthwise Rocking	7.4	7.13	6.67
Lengthwise Rocking	9.25	9.75	
Edgewise Rocking	11.1	11.67	10.2
1st Bending	14.5	14.1 Pure Bending 14.5 Bending/Rocking	12.4
1/1 "Plate" Mode	**19.85** **Motor = 20 Hz**	**19.2**	**19.0**
2nd Bending	28.8	28.2	28.5

Table 9. Natural Frequencies of the Full-sized Table, Compared With Scale Model and Computer Model Predictions **Without Machinery in Place. (**Later testing, <u>with</u> machinery, found the "bounce" mode was 5.4 Hz.)

We confirmed that:

1. Concrete has lower damping than most people think, and it can be accurately modeled using aluminum.

2. The concrete slab constantly changed shape during the year, bowing up and bowing down, as the temperature of the outside air changed. The top surface of the table was enclosed by a large "Butler Bldg." which also contained the control room (CCR), so the top was always warm. Since all of the piping went down, the bottom of the table was exposed to hot summer and cold winter temperatures and the temperature of the machinery and pipes. For the first year, or a little longer, we measured the "gap" between the edge of the spring bucket and the bottom of the cap piece using an "inside micrometer" at 4 locations 90° apart (later reduced to two) per spring. The data were reported to the plant and the compressor manufacturer. I no longer have any of the data, but I don't think alignment ever became an issue. As far as I know, no changes to the design were considered. The concrete foundation and the springs have performed well since startup in 1980.

8.6 SOMETHING TO THINK ABOUT WHEN CHOOSING A MATERIAL FOR THE MODEL

Why is the model aluminum? The velocity of sound in concrete is 11,823 ft/sec, and the damping is low enough (2% to 5% of critical) that both steel and aluminum can be used to model concrete structures. The aluminum model was tested in our laboratory, and it predicted the natural frequencies of the full-size slab, as we later learned, with about the same accuracy as the best computer model did!

The first model slab we built was $1/19^{th}$ scale ($\alpha = 19$) and was constructed out of concrete reinforced with "rabbit wire." However, during handling, it was dropped and cracked all the way through. It was discarded. The aluminum model was built because it was more rugged than the concrete model and we could hold dimensions better. The second model milled out of plate stock and weighed 130 lbs. This model, 1/23rd in size ($\alpha = 23$), took five (5) days to machine.

9.0 CONCLUSIONS & WHAT NEXT?

1. "Equation 1a" provides remarkably accurate prediction of the "natural frequencies" for the "modes of

$$f_{n\ new\ i} = (f_{n\ old\ i} / \alpha) (C_{vv\ new} / C_{v\ old}) [(1 - \nu_{old}^2) / (1 - \nu_{new}^2)]^{1/2} \qquad \text{Eq. 1a}$$

   vibration" of machinery or structures, before the new machinery or structures are built, and in many cases, before a computer model is "built," for almost any combination of "materials of construction."

2. "Equation 1" is every bit as relevant and accurate today as it was 40 years for cases where both materials have the same Poisson's ratio.

3. Use the <u>calculated</u> value for "velocity of sound in a material," if you can:

$$C_v' = \sqrt{E / \rho} \quad \text{in/sec}, \qquad \text{Eq. 2}$$

   then

$$C_v = C_v' / 12 \quad \text{ft/sec.}$$

   Measured values never seem to fit the calculated values, and several materials have C_v values very close to one another. Measured values have the chance of experimental error, or waves that are not pure shear or compression. If you have to use measured data, make sure it is from the same source for both materials.

4. Care must be taken when predicting materials with different Poisson's ratio, which technically "distorts" a "true" scale model (one where all materials and dimensions are changed the same way, such as the model "pump rotors" in Section 4.3), and if one tries to apply "metal" to plastics, then you are destined to fail. (See Appendix D).

5. Natural Frequency & Mode Shape Of Vibration – New Definitions offered by the author:

   A "natural frequency" has two parts:

   - the "frequency" of vibration
   - and an associated "mode shape", or "pattern," of vibration.

The “frequency” of free vibration is determined by the <u>material properties</u>, those being the “velocity of sound in the material” and “Poisson’s ratio,” and

The “mode shapes of vibration” of a machine component or structure is determined <u>solely by the geometry</u> of the machine component or structure.

6. Less than a week after this paper was submitted for publication, a paper [10] was found that seemed to validate our conclusions in 1978 [1]. Coming from “wave theory” (we came from “beam” and “plate” theory), a Section on “Wave reflection from various geometric boundaries arrives at equations for “transmitted” wave and “reflected wave” in terms of:

$$\sqrt{E_2 \rho_2 / E_1 \rho_1}$$

which are the variables that make up velocity of sound in a material, but not stacked the same way.

On page 4 of Reference [10], for “Natural frequencies and resonance” they write: **“The material properties determine the wave velocity, and the geometry determines how waves are reflected and refracted**.” Sounds like **our conclusion exactly**.

APPENDIX A

BACKGROUND – LIFE IN THE 1970'S

"Equation 1" came about as a result of a series of "unfortunate events," and a subsequent research project aimed at finding a way to make sure that didn't happen again. The "events" were:

- A project to create a "permanent" impeller for a particular waste well pump was a "complete and abject failure."

- Several other Company projects needed months to complete start-up of new one-of-a-kind machinery and process equipment, costing time for re-engineering components and ironing out resonance-related vibration problems.

- New (in 1975) finite-element computer software, such as NASTRAN, claimed to be able to do dynamic analysis for complex machinery components, but they needed expensive mainframe computers and large amounts of expensive computer time, and they were not very accurate.

- As test engineers, we needed a physical object to test, so how do you create a model early? And we had a lack of experience with "physical" models ("scale" models are just one of four subsets of the category called: "physical" models). What was the best approach?

- Frustration on the part of our mechanical engineers that we had not been able to get our arms around the predictive questions we were being asked to answer.

APPENDIX B.

1. WHAT IS A “NATURAL FREQUENCY, AND WHY IS IT SO IMPORTANT?

Here is an easy way to get a feel for resonant vibration. Get a heavy weight, like a big accelerometer magnet, and connect it to a large, fairly soft rubber band. Hold the rubber band so the weight is hanging from it (Figure B.1).

Now move your fingers up and down very slowly. Watch the weight. It moves 1:1 with your fingers. If your fingers go 1 inch up, the weight goes 1 inch up. This is “**forced response**.” The output is equal in amplitude to the input.

Next, start moving your fingers up and down faster. Soon you will reach a frequency where your fingers are barely moving, but the weight is bouncing several inches up and down. This is the “natural frequency” ($\omega = \omega_n$) for this weight (W) on this rubber band (K). You might be moving your fingertips up and down 1/16th inch (pk-pk) and the weight is moving 2 inches (pk-pk) up and down. The input frequency equals the natural frequency, and you are getting large output response for a small amount of input excitation. This is “**resonant” vibration,**” or “**resonance.” “Resonance magnification**” is the ratio of how much greater the “output response” is to the amplitude of the input. The “magnification factor” is the ratio of output over input. In this example, magnification is 2 in p-p / 1/16 in p-p = 32:1.

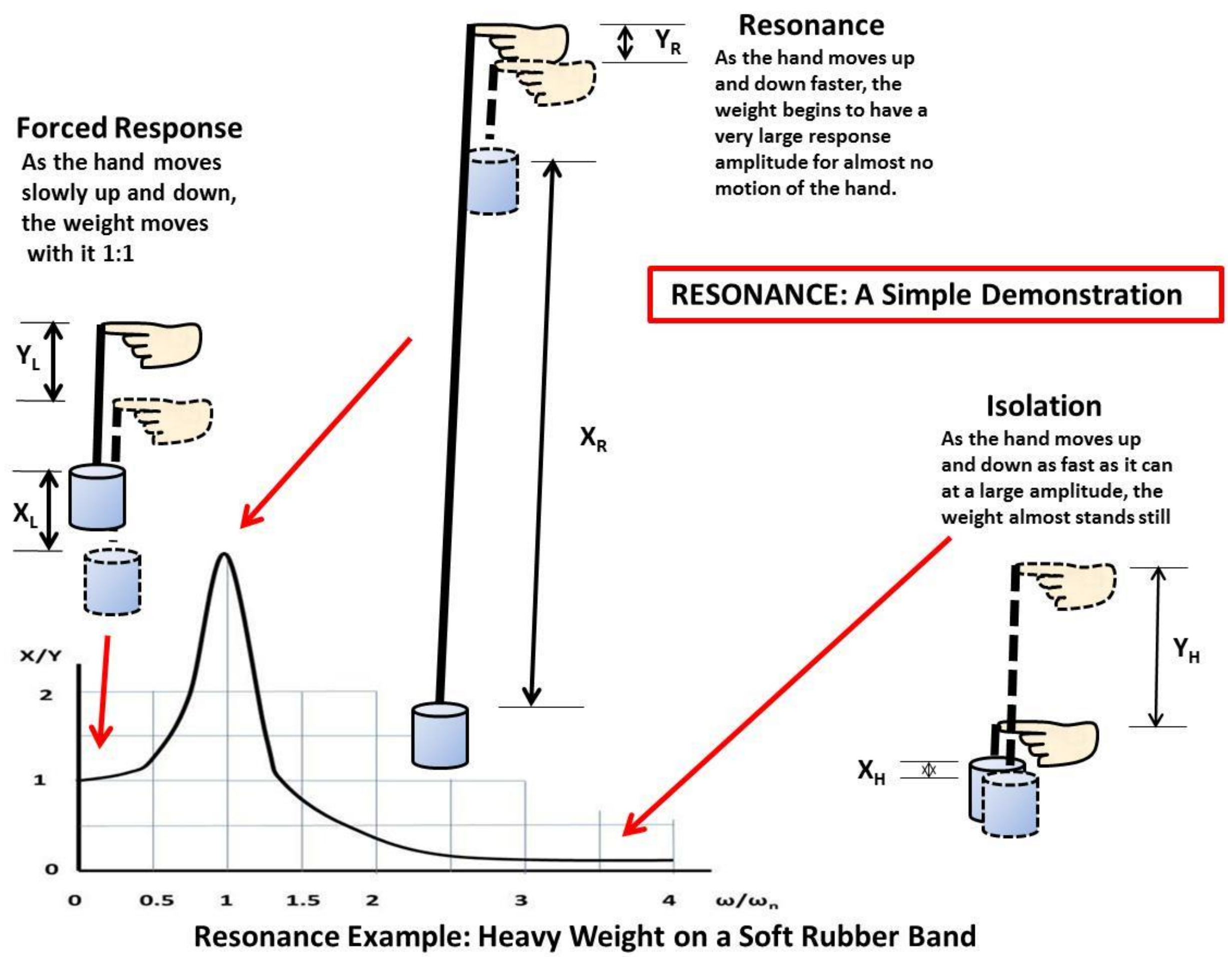


Figure B.1 A simple demonstration of “**Resonance”**

Next start moving your fingers up and down as fast as you can. The weight will almost stand still, while your fingers are moving up and down maybe an inch at high speed. This is the phenomenon of "**isolation**." If the "K" of the system that weighs "W" is selected so that the natural frequency, ω_n, is 1/3rd to 1/4th of the operating speed, ω, typically only 10% to 20% of the input force will be transmitted to the weight. So the weight barely moves. This is vibration, or force, "isolation" and it is a very important subject for a young engineer to understand if, like I was, that young engineer is asked to isolate extremely sensitive instruments from floor vibration. Such instruments are "NMRs" ("nuclear magnetic resonators") and "AFMs" ("atomic force microscopes").

A question is: Why did the amplitude of the motion of the weight on the rubber band not go to infinity when the frequency went through the resonance frequency? The weight did not even hit the ceiling. The answers are – damping and time:

1. **Damping** –"Extensional damping" due to deformation of the rubber band limited amplitude at resonance. To a lesser degree, the air ("viscous damping") helped damp the amplitude, as well. But the damping only made a difference at the resonance speed. See Figure B.2 from Ref 5 below:

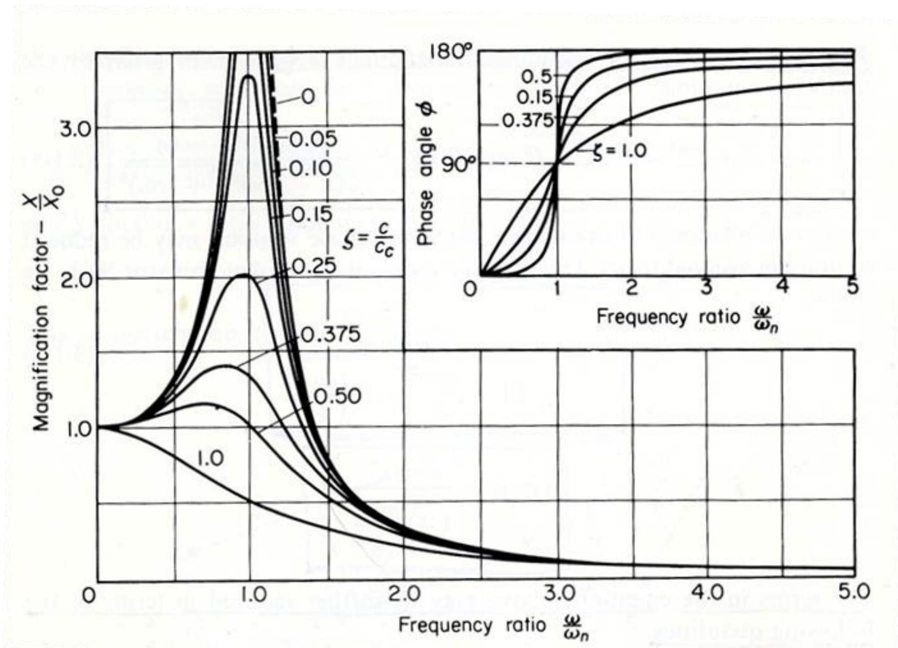


Figure B.2. Resonance amplitude depends on damping [8]. Viscous damping ratios from 0.05 (light damping) to 1.0 ("critical" damping) are plotted here. Amplitude also depends upon rate of acceleration through the critical speed frequency range.

Damping is only effective at or near resonant frequencies. "Resonance" is when $\omega / \omega_o = 1$. At frequency ratios below 0.75 and above 1.25, in Figure B.2, the amplitude is the same regardless of the damping. At frequencies above resonance, damping can actually cause vibration to be higher than if damping was less. That was a problem with early "plastic valve covers" in cars. Marketing thought plastic had more damping than metal and would lessen the valve noise across the frequency range. But plastic doesn't really have much damping. And what it does have only works on portions of the frequency range in which the cover has natural frequencies. At the other engine speed ranges, damping of the plastic cover wasn't helping. Marketing was disappointed.

2. **Time** – Increasing the finger speed very quickly with the weight on the rubber band does not allow time to build up amplitude. This is very important when you are trying to get a lightly damped rotor through a critical speed. Park the rotor at a speed below the critical, then increase speed very quickly. The rotor will not typically have time to build up a large amplitude. However, you need to know where the critical speed is before you do this. If you don't, you might increase speed and park it on a critical and bow the rotor.

Damping is **energy dissipation,** usually in the form of heat energy, but other forms include noise and material deformation. Damping limits amplitudes of motion. The major types of damping I normally encounter are:

1. V**iscous Damping**

This is like a boat moving through water, or a shock absorber on the suspension system of a car. The water resists the motion of the boat. Force is required to move the water out of the way for the boat to proceed. Cut the engine off in a ski boat and the boat will immediately slow down and come to a stop.

2. **Solid, or Structural Damping**

   This is "hysteretic" (or "material") damping. Energy is dissipated through deformation of the material. In a simple tensile test of a polyethylene plastic, enough heat can be generated as the plastic stretches that the plastic actually melts during the test! The heat is being generated faster than it can be dissipated. See Figure 11 of this paper.

3. **Coulomb Friction**

   Everyone is familiar with "friction." This is the primary damping mechanism in bolted joints.

Another I run into from time to time is --

4. **Active Damping -- Using Controllers On Actuators To Invert Vibration Signals, So The Vibration Cancels Itself, Producing Ultra-quiet Vibration Tables For Electron-Beam, Micro-Chip Manufacturing Systems**

   This is "active" vibration cancelation, where the motion is electronically measured, and then turned on itself. This is used to limit vibration of laboratory isolation tables to incredibly small amplitudes, including "MEBES" systems.

5. **Other**

   There are others. The author recalls a list of seven, but cannot remember them all. These are the big four for the type of mechanical work I have done in the rotating machinery and chemical plant business.

APPENDIX C

WHAT IS "VELOCITY OF SOUND" IN A MATERIAL OF CONSTRUCTION, AND WHY IS IT"

The "velocity of sound" is the speed at which a "stress wave" or an "impulse" travels in a material.

One common method of determining the "velocity of sound in a material of construction" is to strike a cylindrical sample, like a rod, of a material on one end, and measure the time required for it to reach the other end. The velocity of the wave is the length of the rod divided by the time it took for the pulse wave to reach the other end. That is the measurement method.

The method used in this paper, with which I have had good experience, is to calculate the "velocity of sound" from Equation 2, knowing the "elastic modulus" (lbf/in^2) and the mass density (lbf-sec^2 / in ^{4}) of the material. Once the material properties have been determined for a particular material, there is no measurement error to worry about, the velocity at elevated temperatures is available, and we don't have to really have the material in hand to consider it for use., The specific weight, itself, is calculated by dividing the weight (lbf) by the "volume" (in^3) of the sample . An example:

Armco 15-5 PH VAC CE
Precipitation-Hardening Stainless Steel **Poisson's Ratio = 0.272**

Condition		A	H 900	H1075	H1150
Density	lbf / in3	0.281	0.282	0.282	0.283
	(gm/cm3)	7.78	7.80	7.81	7.82
Modulus of Elasticity					
	psi		28.5E+6		
	(MPa)		(196E+3)		

Young's modulus = 28,500,000 lbf/in^2

specific weight, γ = 0. 282 to 0.283 l lbf/in^3

mass density, $\rho = \gamma / g_C$ = 0.282 lbf/in3 / 386 in/sec^2 = 0.00073057 lbf-sec^2 / in^4

Then

Armco 15-5 PH VAC CE
Precipitation-Hardening Stainless Steel **Poisson's Ratio = 0.272**

Velocity of Sound <u>in</u> Armco 15-5 PH VAC CE , Precipitation-Hardening Stainless Steel:

a. H-1075 at 70° F (21° C) with 0.282 lbf/in^3--

$$C_v = \sqrt{E / \rho} = (28{,}500{,}000 \ / \ 0.0007306)^{1/2} \qquad ((\text{lbf} / \text{in}^2) / (\text{lbf-sec}^2 / \text{in}^4))^{1/2}$$

Noting

$$C_v = (3.9009 \text{ E+10})^{1/2} \qquad \text{and} \quad ((\text{ lbf} / \text{in}^2) \times (\text{in}^4 / \text{lbf-sec}^2))^{1/2} = (\text{in}^2 / \text{sec}^2)^{1/2}$$

or

$$C_v = 197{,}507 \text{ in /sec} = 16{,}459 \text{ ft/sec}$$

b. H-1075 at 600° F (316° C) with 0.282 lbf/in^3 and 91.4 % of the Elastic Modulus

$$C_v = \sqrt{E / \rho} = (0.914)(\, 28{,}500{,}000 \;/\; 0.0007306\,)^{1/2}$$

$$C_v = (\, 3.5656\text{ E+10}\,)^{1/2}$$

$$C_v = 188{,}827 \text{ in/sec} = 15{,}736 \text{ ft/sec}$$

c. H-1100 at 70° F (21° C) with 0.283 lbf/in^3--

$$C_v = \sqrt{E / \rho} = (\, 28{,}500{,}000 \;/\; 0.00073316\,)^{1/2} \qquad ((\text{lbf}/\text{in}^2)/(\text{lbf-sec}^2/\text{in}^4))^{1/2}$$

Noting

$$C_v = (\, 3.8873\text{ E+10}\,)^{1/2} \quad \text{and} \quad ((\text{lbf}/\text{in}^2) \times (\text{in}^4/\text{lbf-sec}^2))^{1/2} = (\text{in}^2/\text{sec}^2)^{1/2}$$

or

$$C_v = 197{,}161 \text{ in/sec} = 16{,}430 \text{ ft/sec}$$

d. H-1100 at 600° F (316° C) with 0.283 lbf/in^3--

$$C_v = \sqrt{E / \rho} = (\, 28{,}500{,}000 \;/\; 0.00073316\,)^{1/2} \qquad ((\text{lbf}/\text{in}^2)/(\text{lbf-sec}^2/\text{in}^4))^{1/2}$$

Noting

$$C_v = (\, 3.553\text{ E+10}\,)^{1/2} \quad \text{and} \quad ((\text{lbf}/\text{in}^2) \times (\text{in}^4/\text{lbf-sec}^2))^{1/2} = (\text{in}^2/\text{sec}^2)^{1/2}$$

or

$$C_v = 188{,}493 \text{ in/sec} = 15{,}708 \text{ ft/sec}$$

Another Method For the Determination the "Velocity of Sound" in a Material

The "velocity of sound" is defined in "Equation (2)." The sketch in Figure 12 shows how one can determine Young's modulus, "E." A small sample of the material in the form of a plate can be used to determine the "elastic modulus" by clamping the sample to create a cantilever beam and "twanging" it to get the 1st natural frequency (in Hz). This frequency is substituted into Equation 3, which is the cantilever beam natural frequency equation, rearranged to determine "E."

If you need to do tests on the material to determine C_v, try to have the two samples tested with the same tester at the same time. Tests done at different times on different machines can have differences in the results that are not associated with the value of C_v. Using the calculated values of C_v eliminates measurement errors and discussion of shear versus compression wave.

Problems Found With Calculating the "Velocity of Sound in a Material of Construction

Values found in the literature for "specific weight" or "density" are usually consistent, but the values for Young's modulus, or the "elastic modulus" can vary based on temperature, type of heat treat, etc. Example: Young's modulus for "mild carbon" steel -- Is it 29.0E+06 psi, 29.5E+06, 30.0E+06, or another number"?

In this case, the author opts for the mid-range value," and "Velocity of Sound," C_v, for:

"Mild" carbon steel E = 29.5E+06 psi Sp Wt = 0.283 lbf/in3 Density = 7.3316 lbf - sec^2/ in^4, with the result: C_v = 200,590 in/sec = 16,716 ft/sec = 5,095 meters/sec

"Tantalum 10W," a material used for valves and pipe liners due to its being impervious to almost any acid (except, HF) has a Young's modulus that varies **in the literature** at room temperature from 25.0E+06 up to 30E+06 psi, while the specific weight varies only between 0.603 and 0.608 lbf/in^3. The best approach is to test the material you are going to use. The next best is to look at the nature of the source of the data: A detailed report from the Battelle Institute done for the Defense Department [10] lists "E" for "Tantalum 10W" as "25. E+06 psi" at room temperature, with various other data listed every 100 °F; whereas, a list from a material sales company on the "web" lists it as "30E6" psi. I would put my trust in the Battelle data, which results in the values for 'Tantalum-10W' shown in Table C-1 below:

Material	C_v - Velocity of Sound In Material (ft / sec)	E = Modulus of Elasticity, or Young's Modulus (lbf / in^2)	γ = Specific Weight (lbf / in^3)	ρ = Density of Material (lbf - sec^2 / in^4)
Aluminum	**16,375**	**10E+06**	**0.100**	**2.59E-04**
Brass (70-30)	**11,632**	**15.4E+06**	**0.305**	**7.90E-04**
Concrete	**11,823**	**4.7E+06**	**0.0897** **(155 lbf/ft3)**	**2.32E-04**
Copper	**11,670**	**16.3E+06**	**0.321**	**8.32E-04**
Iron	**16,410**	**28.4E+06**	**0.281**	**7.28E-04**
Paperboard Tubes	**12,945**	**1.574E+06**	**0.0252**	**6.53E-05**
Tantalum-10W	**10,556** **3,200 m/s**	**25.0E+06**	**0.608**	**15.75E-04**
Air, dry, 32°F	**1,088.5**	**N/A**	**4.676E-05**	**1.21E-07**
Water	**4,794**	**N/A**	**0.031**	**8.03E-05**

Table C-1 – Velocity of Sound, Elastic Modulus, & Specific Weight

$$C_v = \sqrt{E / \rho} \quad , \quad \rho = \gamma / g = (\gamma \ \text{lbf / in}^3) / (386 \text{ in / sec}^2)$$

Material	Velocity of Sound in Materials meters per second
Air 40°C	355
Aluminum	6320
Copper	4600
Glass	4540
Gold	3240
Lead	1210
Rubber	60

Table C-2 – Velocity of Sound in other materials in m/s

APPENDIX D

D.0 Plastic Is NOT Metal – If You Use "Metal Logic" In The Design Of A Plastic Part, It Will Fail.

So, IF Plastic Is "NOT Metal" --What Makes Plastic Different From Metal?

D.1 Plastic Is "Viscoelastic"

"Viscoelasticity" – Means having both "viscous" and "elastic" properties. As an example, a shock absorber from an automobile is a viscous and an elastic device. Pull rapidly as hard and fast as you can on the two ends of an automotive shock absorber, and it barely budges. Your hands go nowhere.

Next, try the opposite: Pull your hands **slowly** and **smoothly** in opposite directions. The shock absorber now extends in length easily and smoothly. Your hands glide apart until internal pistons hit the "stops."

Below the yield stress, **metal** is "**elastic."** Stress is a linear function of strain. The slope of the stress/strain curve is "Young's Modulus," also known as the "elastic" modulus. Take the load off and strain returns to zero.

Plastic is "**viscoelastic."** Stress in a viscoelastic material is a function of strain and **time**. Was the time period seconds, minutes, or hours? It makes a big difference. Under stress, plastic "**creeps," "recovers,"** and **"relaxes**" (Figure D.2). Take the load off and strain may not return to zero, or at least not right away.

D.2 Plastic Does Not Conduct Heat Like Metal

When a material is stretched, or bent, or twisted, or displaced back-and-forth cyclically, like in a "fatigue" test, tiny little molecules of material get distorted and dragged over other tiny little molecules of the material. That generates **heat. In metal, the heat is conducted away**, but plastic does not conduct heat like metal. Plastic can actually **melt** during a standard tensile test, as the tensile sample in Figure D.1 did, for example.

Figure D.1 A "tensile" test sample cut from the flat, side panel of a large plastic shipping an container suddenly **melted** during the test. The shipping box was molded from "high-density polyethylene" (HDPE) using the "foamed-core" process. This is the same sample as the one shown in the photograph in Figure 11. It is shown larger here to better see the melted zone.
0.2 inches per minute:

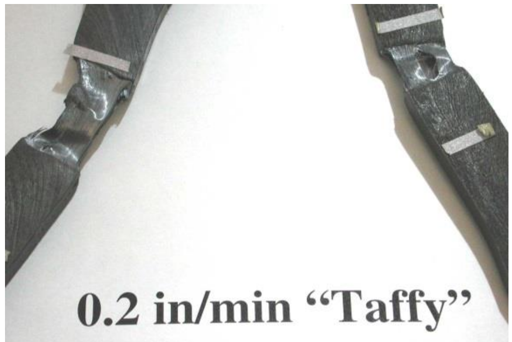


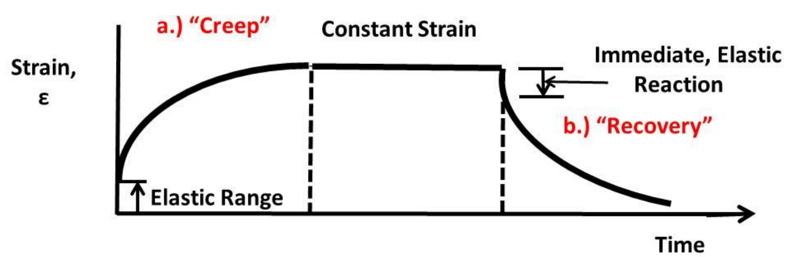
Figure D.2 All five samples tested at 0.2 in/min melted. These are two of the five.

When you stretch a sample, metal or plastic, in a standard "tensile test," the sample generates internal heat. If you stretch "foamed polyethylene" at exactly the "critical" strain rate -- it melts! Figure D.2 shows two more of the five samples of "foamed HDPE" mentioned at the beginning of this Appendix. The plastic generates internal heat faster than it can conduct it away. An MIT Professor called it the "Davindekov point," when I showed her the samples.

D.3 The Three "Behaviors" of Plastic – Creep, Recovery & Relaxation

If you pull on a tensile test sample, as illustrated in Fig. D.3.a. you have a **constant stress** situation, and if the plastic continues to elongate under that same stress, that is "**creep**." If, as in D.3.b.), the **stress is suddenly removed**, there is an instantaneous "**elastic recovery**" – but not all the way to zero – followed then by a gradual "**viscous recovery**" with time. And it might take a long time to get back to near zero strain, or it may never return to zero.

D.3.a



D.3.b

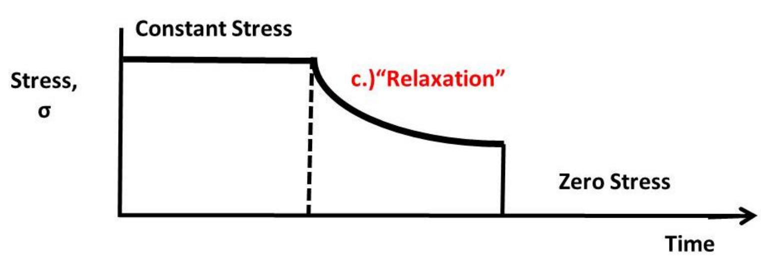


Figure D.3. Three "behaviors" of plastic. These behaviors complicate strain gage interpretation.

In Figure D.3., a "**constant strain**" situation has been established. The part, however, can experience "**stress relaxation**." Place a cube of plastic between two metal plates, and push down hard on the top plate through a load cell (which measures force) squeezing the plastic cube. Then hold the plates that same distance apart -- and wait for a few hours. The **strain gage** on the part will show the **strain** has not changed. But the load cell will show the force is less than was initially required to achieve that strain. That is, the **stress** is reducing with time, or it is **“stress relaxing.”** Eventually, the part will become loose. This is not creep, because the plates were locked a constant distance apart – the external deformation of the plastic cube is the same. The plastic part is just not pushing back **as hard** against the metal plates. It is apparently internally rearranging itself to reduce stress. This is the “**visco-“**in: **“viscoelastic stress relaxation.”**

D.4. Other characteristics the designer needs to know:

This is just a small part of a much longer list.

- The **elongation** of plastics can be a problem. In metals, the maximum strain range is on the order of 0.001000 in/in, or 0.1%. In plastics, even glass-filled nylon can exceed 1% and DuPont Zytel™ 103 (almost pure nylon) can exceed 50% strain. That is 500 times greater than metal

- The “**flexural modulus**” (essentially the “elastic modulus”) for plastic is far less than the elastic modulus for metal -- 500,000 to 1,000,000 psi for glass-filled nylon versus 10,000,000 for aluminum and 30,000,000 psi for steel.

- The **fracture surface** can be different depending on how fast the fracture occurred – and then, sometimes not.

- Plastic can fail at zero strain rate – that is, when a constant stress is applied over a sufficiently long period of time. This is called “**creep failure,” “creep rupture,”** or “**static fatigue**” [17].

- The standard test for **fatigue data** is conducted at an alternating frequency of **1800 cpm** (30Hz) due to the just-mentioned heat generation problem.

- Plastic has **damping**, but less than people think. It is easy to measure using the “half power” method, but a few “tricks” can improve the accuracy of the value determined. And you should **avoid** using the “log decrement” method. It is difficult if more than one mode is excited.

D.5 Load-Deflection Curves, “Flexural” Modulus and Other Plastics Test Data

The plot in Figure D.3 is the “manufacturer’s catalog” stress/strain diagram for a glass-filled nylon (an “engineering plastic”) at four temperatures (-40˚C, 23˚C, 93˚C, and 149˚C) and at two moisture conditions (dry as-molded and 50% RH). As you can see, both temperature and moisture absorption make big differences in the stress-strain relation. Other differences you might have noted are:

- The strain range is 0 to 10 %. “10% strain” is 0.100000 inches/inch. In steel, that would be 3,000,000 psi. Here, the stress scale is 0 to 240 MPa, or 0 to 34,800 psi. There’s that 500:1 to 1,000:1 ratio I mentioned earlier.

- The highest temperature reported is 149˚C (300˚F). This is because nylon is an **amorphous** plastic, with a “**glass transition temperature**, $\mathbf{T_g}$, below which the material acts like glass; i.e., it is hard and rigid” [17].

In the diagram in Figure D.4., “flexural” modulus is plotted versus temperature. The “flex” modulus for plastic is sort of the equivalent of Young’s modulus (or, the “elastic” modulus) for metals.

D.6 Plastic Models

There are at least **four different models** which try to simulate the viscoelastic nature of plastics [17]. They are:

- The “Maxwell” model,
- the “Kelvin” or “Voigt” model,
- the “Maxwell” and “Kelvin” models in series, and

- the “Standard Linear Solid.”

The first two models are simple. The last two are a little more complicated. All four are covered in detail, including equations, in the book: **PLASTICS ENGINEERING, 2nd Ed**., by R.C. Crawford, Chapter 2 [17]. It is the best book I have found on the “behaviours” of various plastics. And it’s a book from the UK, hence, the “u” in “behaviors.” If you are interested in the equations for these models, and an explanation for the fourth model, the “Standard Linear Solid,” I direct you to Ref. 17, pp. 41 – 73.

D.7 “Creep” and “Recovery”

I think most people can immediately grasp these two behaviors. If you remember the old nylon coat hangers that were around for a few years – the heavy coat that was hung on it a few hours later was on the closet floor. But the coat hanger looked fine. Under the constant load (stress) of the heavy coat, the two “wings” of the coat hanger slowly began to sag down (“creep”) until the coat slid off. Then with the coat gone, the coat hanger wings sprang up and “recovered” their shape in a few minutes. The person was surprised to find the coat on the floor and the hanger looked as if the coat had never been hung on it.

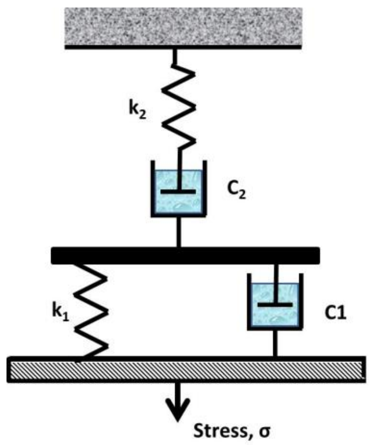


Figure D.4 The “Maxwell” and “Kelvin” models in series. “Maxwell” is above the black bar and “Kelvin” is below the black bar.

D.8 Thermoplastics and Thermosets

There are two types of plastics: a) Thermoplastics, and b) Thermosets. According to R.J. Crawford, PLASTICS ENGINEERING, 2nd Ed. [17], “**thermoplastic**s” are “very long chain molecules held together by relatively weak Van der Waals forces.” When the material is heated, it becomes soft. At a high enough temperature, it again becomes a “viscous melt.” When it is allowed to cool, it solidifies (becomes hard) again. A drawback to using thermoplastics is their properties are heat sensitive. A subdivision of this type of plastic is that some of the plastics are **“amorphous”** and some are **“crystalline.”** I don’t think I have ever needed to delve deeper into the differences between these two categories of plastic to do my consulting job. If it gets deep into the molecular science of plastics, I call the “plastics guys” in that big chemical company for whom I worked. Examples of “thermoplastics” include: polyethylene, PVC, nylon, polypropylene, and polycarbonate.

“**Thermoset**” plastics are “cross-linked” polymers that cannot be softened by the application of heat. If overheated, they will “char and degrade.” Examples include: some of the epoxies and polyesters.

D.9 “Engineering Plastics”

Earlier I used the expression “engineering plastics.” R.J. Crawford says the term probably originated “distinguishing those [plastics] that could be substituted satisfactorily for metal.” Crawford says a more useful differentiation is: “engineering materials” are “able to support loads more or less indefinitely.” He points out that this is a disadvantage versus metals since the plastics have lower strength, and temperature- and time-dependent properties. The advantages, however, are lower density, resistance to chemical attack, ease of processing, and lower cost. Glass fibers are the principal form of reinforcement for plastics, but carbon and Kevlar™, while expensive, can offer additional strengths and stiffnesses. If you need more in-depth information on the "engineering plastics," contact a polymer scientist.

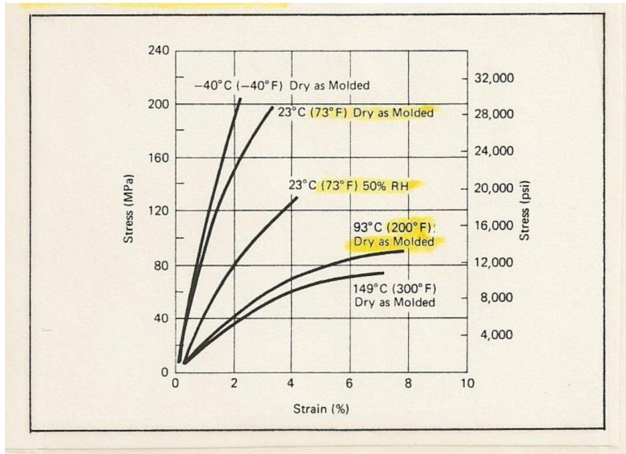


Figure D.5 "Dry as-molded" glass-filled nylon at four temperatures.
Two moisture (RH) conditions are shown for 23˚C (73.4˚F).

In Figure D.6, temperature only goes to 300˚C. Note the big difference between 100% RH nylon and "dry" nylon. Nylon absorbs moisture readily and that makes it weaker. And absorbing moisture also makes the nylon swell. If you are making nylon gears, you have to account for this swelling when cutting the gear teeth and setting the center distances.

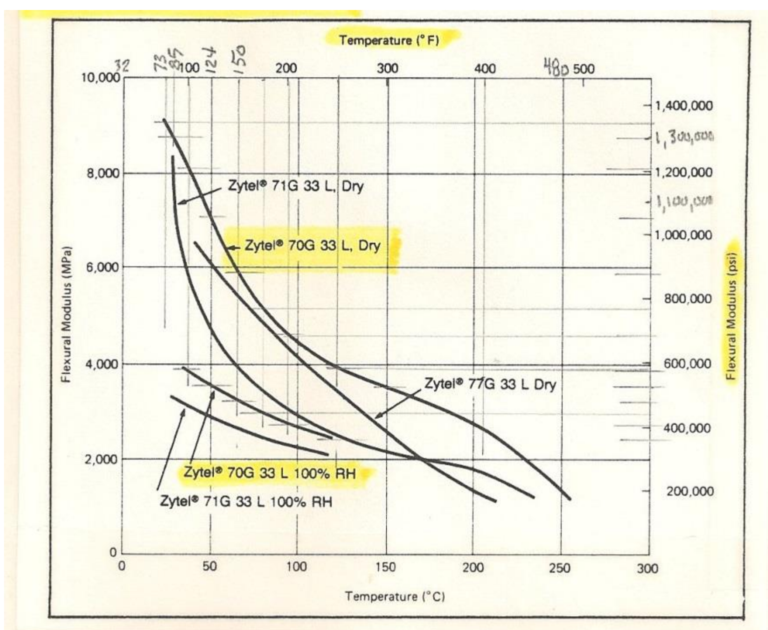


Figure D.6 The "flexural modulus" of 33% glass-filled nylon (Zytel™)
over a broad temperature range.

There is much more that could be presented, but the point has been made that designing with plastic is different than desiging with metal.

APPENDIX E

Results of Literature Search

Literature searches over the years trying to find a better way to predict natural frequencies. We found several similar studies that skirted our work, but they all had different objectives, such as predicting the peak dynamic stress in a moving part by measuring the peak velocity of vibration of the machine [5]. We found clues, such as "Similitude" theory was used in a study [6], but we began our work unarmed except for years of experience and one college course in "Similitude.".

Associated with any large research effort is a search of the literature to assure those paying for the work that it has not already been done somewhere else, to make the research staff aware of any new knowledge on a subject, and to make sure you know what has been tried already and failed (not to be repeated). With the advent of the Web and the various "search engines," searches done by computer can be quickly.

In the technical research areas 35 years ago, one sat for hours in the library combing the "References" list in the closest subject areas to yours, and spent a lot of time documenting titles and figuring out what were the differences between what they did and what you proposed to do. Did their work answer the question, and if not, then what would you do? It was slow and tedious. The writer's PhD literature search and research proposal took 3 months, as an example. With that background, and having done a stellar "literature search myself and knowing how important it can be …35 to 40 years ago we found two research programs that mentioned the subject of ways to "predict natural frequencies and mode shape changes when neither contributed much to our program" and both changed direction.

The amazing accuracy of "Equations 1 and 1a" in predicting literally dozens of vibration resonance frequencies for mode shapes in a sequence as shown in Tables 3 and Table 8 has proven its value over the years, but to be that close and not have the chance to go into depth on predicting dynamic stress in a resonance situation was a bit frustrating. Even today, dynamic stress prediction using finite-element computer modeling techniques on complex structures such as fan wheels is still limited to the most fundamental two or three modes as far as accuracy of prediction of dynamic stresses is concerned. Measurements taken on a scale model even today, would be more accurate than a computer model in predicting the modes and stresses in a bolted sieve tray or in the vanes of a complex centrifugal fan wheel.

"The equation" is derived which predicts the natural frequency of an object, machine component, etc., made from a new material from the natural frequency of the original object, machine component, etc., and the "velocity of sound" in the new and the original materials. "The equation" is:

$$\boldsymbol{\omega_{i\text{-}P} = (\omega_{i\text{-}m} / \alpha)(C_{v\text{-}p} / C_{v\text{-}m})} \qquad \text{Eq. 1}$$

It says -- if you know the "velocity of sound" (C_v) in the two materials, when the discussion of changing the natural frequency by changing the material comes up, you can tell them immediately what the new natural frequency will be.

1. "Dynamic Stress And Velocity In Resonance Vibration," Stephen H. Crandall, MIT, 1968 [5]

Developing the relation between stress and velocity in resonant vibration had been a goal of experts such as Stephen H. Crandall, MIT, [5] and others in the 1960's. The tray-cracking problem we tackled, described previously in Section 2.10, had all the elements needed for a thorough research study of stress and velocity in resonance; however, the research direction in our laboratory shifted and funding for that type effort was drained by higher profile programs.

I still have not seen this equation in the literature (other than what I wrote), but I did come close. Well after we had proven the capability of the equation on distillation column sieve trays (Section 7.0), I came across a "Letter to the Editor" of The Journal of the Acoustical Society of America written by Stephen H. Crandall, MIT Professor, on the subject of the "Relation between Strain and Velocity in Resonant Vibration." Ref. [5] It had been published in December 1962.

In the letter, Crandall observed that dimensional analysis "indicates that for a particular mode of free undamped vibration the natural frequency ω_n satisfies a relation of the form

$$\omega_n L / c = k_1 (\text{shape}, \nu) \qquad \text{Eq. E2}$$

where $c = (E / \rho)^{1/2}$ is the velocity of sound (for longitudinal waves in a slender rod)." [5]

Crandall was writing in response to a paper published on "Stress and Strain Limits on the Attainable Velocity in Mechanical Vibrations" by F.V. Hunt [9]. Specifically Crandall was discussing the "proportionality constant "k" in the equation

$$\varepsilon = k (\nu / c) \quad , \qquad \text{Eq. E3}$$

where "ν / c" is the "vibrational Mach number" and ν is the "maximum vibratory velocity."

For our work this was an academic exercise. Our tests on full and half-sized stainless steel and aluminum sieve trays had already demonstrated the equation worked for vibration and dynamic stresses in low-damped structures with multiple resonances. Crandall's Equation 2 is presented for completeness, since it is very close in form and identical in purpose to our work. We just happened to find the paper in a literature search conducted as part of our documenting several years of similitude research.

What was more interesting about Crandall's paper, and a paper by M.C. Plummer in 1978 [12], was relating the maximum strain in a vibrating part to the maximum vibration of the part. We were moving into machinery condition monitoring research and thought the idea of making a vibration reading on a machine part in the field and on a scale model part somewhere else, and converting that to the maximum strain on a part inside the machine, was the type of new technology we needed to study.

The machinery users were already doing periodic monitoring following T.C. Rathbones classic paper [8], but they were looking at vibration displacement. Alarm levels were different for different frequencies. Plummer suggested changing to vibration velocity, pointing out that:

- Velocity was a more "sensitive indicator of machinery condition,"

- Velocity alarm levels would be "independent of the operating speed" of the machine, and

since the maximum vibrational stress, S, had been shown to be proportional to vibration velocity, $\pi f V$, it followed that

- vibrational stresses in geometrically scaled systems made of the same material could be compared by measuring the vibration velocity at corresponding points.

The last point about comparing vibration stresses and velocities matched where we were going exactly. Plummer [10] equated maximum kinetic energy (maximum velocity) to maximum strain energy (zero velocity), and came up with an equation for "maximum vibrational stress, S:

$$S = (\pi f V) \sqrt{K1 / K2} \sqrt{E \rho} \qquad \text{Eq. E4}$$

where K1 and K2 are constants related to the deflection curve and mass points on it. We did not pursue this, since we had developed our own equation expanding on another paper which related "dynamic stress" in a scale model to dynamic stress in a piece of actual mechanical equipment. **"Dynamic stress monitoring" for mode "i"** used the following relationship:

$$\sigma_i{}^*{}_{p\text{-}max} = \sigma_i{}^*{}_{m\text{-}max} (E_p / E_m) (C_{vm} / C_{vp})(V_{max\text{-}p} / V_{max\text{-}m}) \qquad \text{Eq. E5}$$

The equation made sense to us. I remember developing it based on an equation in another paper (which I cannot find), but work direction changed and we never had time to explore this in any depth. It is presented here in case anyone has an interest in exploring this machine component monitoring approach. We offer no guarantee that it is accurate.

2. Author unknown, "Vibration theory," Soc Petrolum Engg, http//petrowiki.org/Vibration theory, circa 2015, 7 pp.

Less than a week after this paper was submitted for publication, a paper was found that seemed to validate our conclusions drawn as far back as 1975. Coming from "wave theory" (we came from "beam" and "plate deflection" theory), a Section on "Wave reflection from various geometric boundaries arrives at equations for "transmitted" wave and "reflected wave" in terms of

$$\sqrt{E_2 \rho_2 / E_1 \rho_1} \qquad \text{Eq. E6}$$

that define the "velocity of sound in a material," but not arranged that way.

On page 4, for "Natural frequencies and resonance" they write: **"The material properties determine the wave velocity, and the geometry determines how waves are reflected and refracted**." Sounds like our conclusion exactly.

APPENDIX F --

Springs Made From "Belleville" Spring Washers

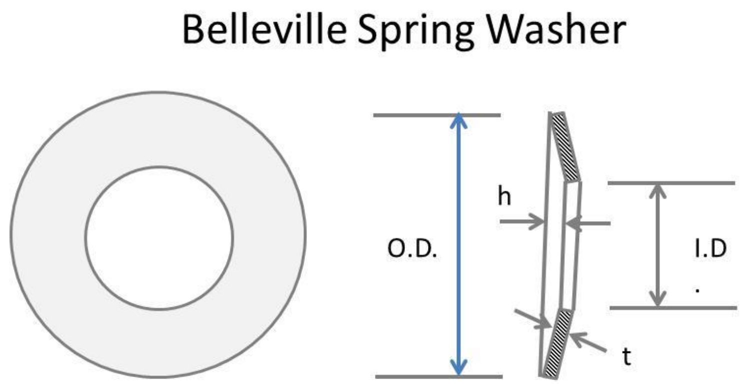


Figure F-1. A single Belleville washer or spring.

The equation for the "spring constant," or the "stiffness," is complicated and non-linear for height-to-thickness ratios, h / t, greater than 0.4 (Ref. 9):

$$k = W / \Delta$$

$$= (E / (1- \nu^2) M a^2) ((h - \Delta/2) (h - \Delta) \quad + t^3) \qquad \text{Eq. F1}$$

where

k = stiffness, lbf / inch, is within 2.5% of linear for $h / t = 0.4$

W = load, lbf

Δ = deflection, inches

E = Young's Modulus, lbf / in^2

ν = Poisson's ratio, 0.3 for steel

a = ½ the O.D. , inches,

and α = the ratio of O.D. / I.D.

$M = (6 / \pi \log_e \alpha) ((\alpha -1)^2 / \alpha^2)$, or the chart of Figure 43 of Ref. 9.

The individual "coned-disk" springs can be rearranged to alter the overall "effective spring rate," K, of the spring assembly. A few combinations are shown in this section

With four "Belleville" washers in a can, by my count, there are nine (9) different load arrangements with "overall spring stiffness" value, K, ranging from 0.25k , 0.33k, 0.4k , 0.5k , 0.67 k, 1k, 2k, 3k, and 4k, where "k" is the stiffness of one "Belleville" spring, by itself. If a spring washer is removed from a can, it would be replaced by a flat, circular steel "spacer" plate. Five of these nine arrangements are shown in Figure F-2.

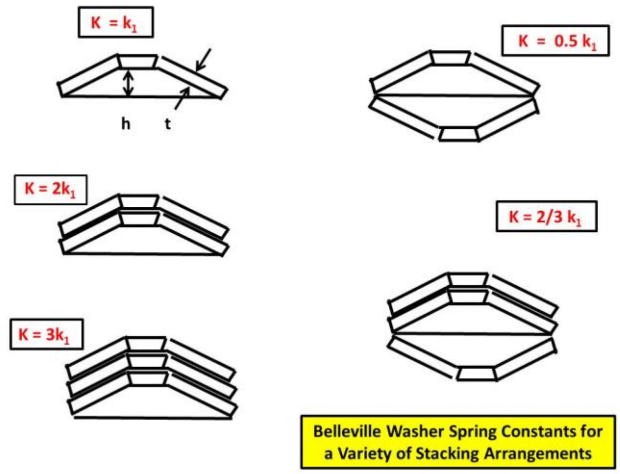


Figure F-2. Some "load arrangements" for Belleville springs.

<u>How are these "effective spring constants" determined?</u> If "k" is the spring constant for one individual washer, then two stacked in "series" is ½ k, as shown in Figure F-3.

The next arrangement, shown in Figure F-4, probably would only be used where four washers were originally used, and then the stiffness needed to be changed. There would be a round steel spacer plate put into the bottom of the cylindrical spring container. This arrangement is shown here to give an idea of the flexibility of this spring design.

Spring adjustments can be made in a couple of hours. First a screw jack is placed next to the spring, and then the jack carefully raises the slab until the spring container can be pulled out. After removing the top, a couple of the "washers" are flipped over to change the spring rate, the top is put back on, the spring put back in position, and the slab carefully lowered back down onto the spring by the jack – all in all, a pretty neat design.

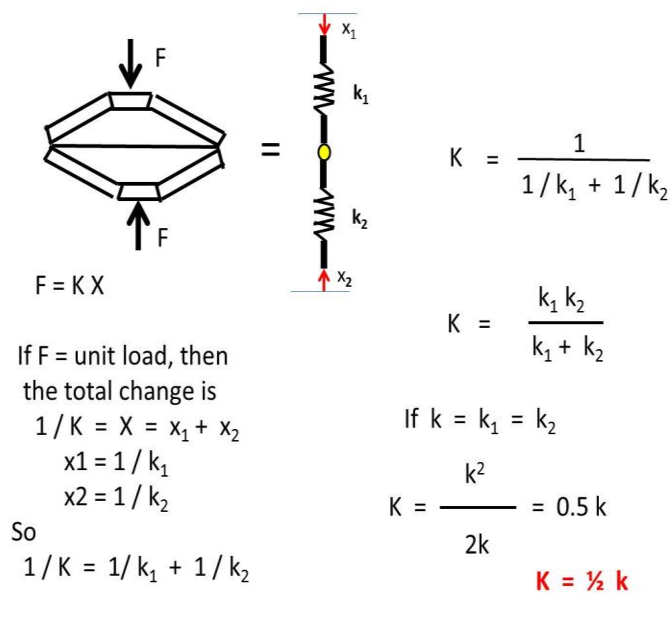
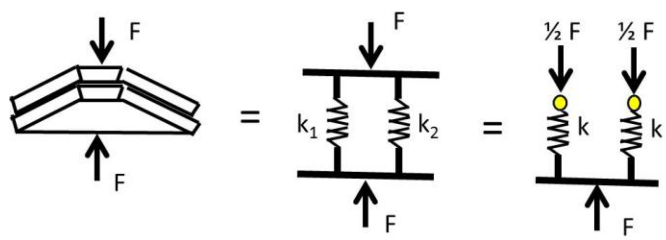


Figure F-3. For two washers in series, $K_{overall} = 1/2k$.



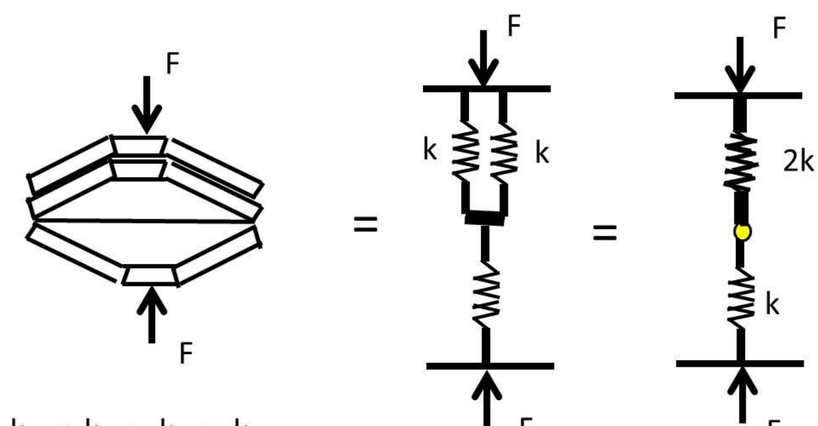
$F = K x$ Where K is the "effective spring rate " for the system

$F = F1 + F2$ But when $k1 = k2 = k$

$F = ½ F + ½ F = k x + k x = (2k) x = K x$

So **$K = 2k$**

Figure F-4. For two washers in parallel, $K_{overall} = 2k$.



$k_1 = k_2 = k_3 = k$

$F = K X$ If F = unit load, then the total change is

$1 / K = X = x_1 + x_2$

$x_1 = 1 / 2k \qquad x_2 = 1 / k$

So

$1 / K = 1/ 2k + 1 / k = (k + 2k) / 2k_2$ and **$K = 2/3 k$**

Figure F-5. An example of the flexibility of the spring design

REFERENCE

1. S.T. Myrick. "Scale Models for Dynamic Analysis of Machinery," DuPont Magazine, DuPont "College Report" (3-page insert) , March/April 1979.

2. Kobayashi, Albert S, HANDBOOK ON EXPERIMENTAL MECHANICS. Society for Experimental Mechanics, Prentice-Hall, Inc., Englewood Cliffs, NJ, 07632, 1987, p. 641.

3. Cyril M. Harris. SHOCK & VIBRATION HANDBOOK, Third Edition, McGraw-Hill Book Company, New York, 1997, Chapter 7, pp. 7-27 thru 7-32.

4. A.W. Leissa. VIBRATION OF PLATES. NASA SP-160, Office of Technology Utilization, Washington, D.C., 1969.

5. Crandall, Stephen H. "Relation Between Strain and Velocity in Resonant Vibration," The Journal of the Acoustical Society of America, Vol. 34, No 12, 1960-1961, Letters to the Editor, December 1962

6. Glenn Murphy, PhD. Similitude in Engineering, The Ronald Press, New York, 1960, pp, 80-82..

7. William T. Thompson, VIBRATION THEORY AND APPLICATIONS. Prentice-Hall, Inc., 2nd Printing, 1965.

8. Rathbone, T.C. "Vibration Tolerances," Power Plant Engineering, 1939.

9. V Hunt, "Stress and Strain Limits on the Attainable Velocity in Mechanical Vibrations." Journal of the Acoustical Society of America, Vol. 34, No 12.

10. Author unknown, "Vibration theory," Soc Petrolum Engg, http//petrowiki.org/Vibration theory, circa 2015, 7 pp.

11 Myrick, S.T, and B.D. Olson, "Dynamic Analysis of Spring-Supported Foundation for 22,000 Hp Electric Motor-driven Compressor." The Chemical Manufacturer's Association (CMA), "Large Electric Motor User's Forum, Chicago, Ill., Nov. 19, 1980.

12. HANDBOOK OF MECHANICAL SPRING DESIGN. Associated Spring Corporation, Bristol, Connecticut, 1964, pp. 70 – 73.

13. "Measurement in the Offshore Industry," Herriott-Watt Univ., Riccarton, Edinburgh, 8 – 11, September 1975.

14. O. Leland, R.D. Begg, and A.C. Mackenzie, "The Dynamic Response of a Fixed, Steel Offshore Oil Platform." Presented at the BSSM/RINA Joint Conference Industry,

15. S Rubin, and R.N. Coppolino, "Sensitivity of Flexibility Monitoring of Offshore Jacket Platforms," Transactions of the ASME, Vol. 108, March 1986, pp 72 – 76.

16. S.T. Myrick, Jr. PhD, L.L.C., SOLUTIONS TO A SPECTRUM OF CLASSIC & UNUSUAL ROTATING MACHINERY & STRUCTURAL DYNAMICS PROBLEMS, From the Memory of Dr. S.T. (Ted) Myrick, Jr., Ph.D., LLC., Begun: 2009 ; Completion expected 2015.

17. R. C. Crawford, PLASTICS ENGINEERING, 2nd Ed., Pergamon Press, New York, 1987.

18. F.F. Schmidt and H.R. Ogden, "The Engineering Properties of Tantalum and Tantalum Alloys," DMIC Report 189, September 13, 1963, Defense Metals Information Center, Battelle Memorial Institute, Columbus 1, Ohio, also: Report AD 426344 [Unclassified], Defense Department Center for Scientific and Technical Information, Alexandria, Virginia, September 1963, Table A-25, page A-49.

CAVEAT

The author and his company, S. T. Myrick, Jr., PhD, LLC, assumes no obligation or liability for any advice furnished by or results obtained with respect to this document. All such advice is given and accepted at the user's risk. This information, based on our experience, is intended for use by persons having technical skill, at their own discretion and risk. We do not guarantee favorable results, and we assume no liability in connection with its use.